CALCULUS II, SUMMER 2015 - EXAM 3

130 points total = 120 points + 10 extra credit points

Name: ______________________________ Score: ______/ 120

N.B. *Do not include scratch-work, but do write neatly and legibly. Everything written on these pages will be graded, so long as it is legible.*

Problem 1 (20 points). Find the radius of convergence of the *hypergeometric series*

$$ {}_2F_1(\alpha,\beta,\gamma;x) = 1 + \sum_{n=1}^{\infty} \frac{(\alpha+n-1)!(\beta+n-1)!(\gamma-1)!}{n!(\alpha-1)!(\beta-1)!(\gamma+n-1)!} x^n, $$

where α, β, and γ are positive integers. Compute $x_2F_1(1,1,2;-x)$, and differentiate $x_2F_1(1,1,2;-x)$ term-by-term to figure out which function $x_2F_1(1,1,2;-x)$ converges to.

Date: August 13, 2015.

Problem 2 (20 points).

(1) Find the Taylor series of
$$f(x) = \frac{1}{3-x}$$
centered at 0, and compute its radius of convergence.

(2) Find the Taylor series of
$$f(x) = \frac{1}{3-x}$$
centered at 2, and compute its radius of convergence.

Problem 3 (20 points). Find the sum of the series

$$\sum_{k=0}^{\infty} \frac{(1.5)_{k+1}}{k!} (-0.75)^k ,$$

where

$$(\alpha)_k = \alpha(\alpha-1)\cdots(\alpha-k+1)$$

is the Pochhammer symbol.

Problem 4 (20 points). Show that the series

$$\sum_{n=1}^{\infty} \frac{\cos(n\pi/3)}{n}$$

converges conditionally.

Problem 5 (20 points). Define

$$a_{2n} = (n!)^{-2/n} \text{ and } a_{2n+1} = \frac{1}{n^2}$$

for each $n \geq 0$. Prove that

$$\lim_{n \to \infty} a_n = 0.$$

Problem 6 (20 points). Compute the fifth-degree Taylor polynomial $T_5(x)$ of $\tan x$, centered at 0. What is $T_5(1)$?

Hint: $\tan x = \frac{\sin x}{\cos x}$.

Bonus Problem (10 points). Recall that the *Fourier cosine series* of an even function $f(x)$ on the interval $[-\pi, \pi]$ is given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx),$$

where

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt)\, dt.$$

Use the fact that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$$

when $f(x) = |x|$ to show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$