CALCULUS II, SUMMER 2015 - EXTRA PROBLEM SET SOLUTIONS

`50 extra credit points`

1. A Smooth, Non-analytic Function (31 points)

Throughout this problem set, we consider the piecewise-defined function

$$f(x) = \begin{cases} e^{-1/x} & \text{if } x > 0; \\ 0 & \text{if } x \leq 0. \end{cases}$$

Problem 1.1 (2 points)**.** Show that

$$\left(\frac{1}{x}\right)^{m+1} \leq \sum_{n=0}^{\infty} \frac{(m+1)!}{n!} \left(\frac{1}{x}\right)^n$$

for each integer $m \geq 0$ and every real number $x > 0$.

Solution. If $n = m+1$, then

$$\frac{(m+1)!}{n!} \left(\frac{1}{x}\right)^n = \left(\frac{1}{x}\right)^{m+1}.$$

Since

$$\frac{(m+1)!}{n!} \left(\frac{1}{x}\right)^n \geq 0$$

for all $n \geq 0$, it follows that

$$\left(\frac{1}{x}\right)^{m+1} \leq \sum_{n=0}^{N} \frac{(m+1)!}{n!} \left(\frac{1}{x}\right)^n$$

for each $N \geq m+1$. Sending $N \to \infty$, we obtain the desired result. ☐

Problem 1.2 (1 point)**.** Use Problem 1.1 to show that

$$\frac{1}{x^m} \leq x \sum_{n=0}^{\infty} \frac{(m+1)!}{n!} \left(\frac{1}{x}\right)^n$$

for each integer $m \geq 0$ and every real number $x > 0$.

Solution. The previous problem shows that

$$\left(\frac{1}{x}\right)^{m+1} \leq \sum_{n=0}^{\infty} \frac{(m+1)!}{n!} \left(\frac{1}{x}\right)^n.$$

The desired result follows by multiplying x to both sides. ☐

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Problem 1.3 (2 points)**.** Use the Taylor series expansion of the exponential function e^x centered at 0 to show that

$$\sum_{n=0}^{\infty} \frac{(m+1)!}{n!} \left(\frac{1}{x}\right)^n = (m+1)!e^{1/x}$$

for each integer $m \geq 0$ and every real number $x > 0$.

Solution. Recall that

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n.$$

From this, it follows that

$$(m+1)!e^{1/x} = (m+1)! \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{1}{x}\right)^n = \sum_{n=0}^{\infty} \frac{(m+1)!}{n!} \left(\frac{1}{x}\right)^n,$$

as desired. □

Problem 1.4 (2 points)**.** Use Problem 1.2 and Problem 1.3 to show that

$$\frac{1}{x^m} \leq x(m+1)!e^{1/x}$$

for each integer $m \geq 0$ and every real number $x > 0$.

Solution. From the previous problems, we have that

$$\frac{1}{x^m} \leq x \sum_{n=0}^{\infty} \frac{(m+1)!}{n!} \left(\frac{1}{x}\right)^n \leq x(m+1)!e^{1/x},$$

as was to be shown. □

Problem 1.5 (3 points)**.** Use Problem 1.4 to show that

$$0 \leq \frac{e^{-1/x}}{x^m} \leq x(m+1)!$$

for each integer $m \geq 0$ and every real number $x > 0$.

Solution. Since

$$\frac{1}{x^m} \leq x(m+1)!e^{1/x},$$

we see that

$$\frac{e^{-1/x}}{x^m} \leq x(m+1)!.$$

Now, $e^{-1/x} > 0$, and the assumption $x > 0$ implies that $x^m > 0$. It follows that

$$\frac{e^{-1/x}}{x^m} > 0.$$

The desired result now follows. □

Problem 1.6 (1 point)**.** Use Problem 1.5 to show that

$$\lim_{x \to 0^+} \frac{e^{-1/x}}{x^m} = 0$$

for each integer $m \geq 0$.

Solution. Fix $m \geq 0$. Since

$$\lim_{x\to 0^+} 0 = \lim_{x\to 0^+} x(m+1)! = 0,$$

it follows from the inequality

$$0 \leq \frac{e^{-1/x}}{x^m} \leq x(m+1)!$$

and the squeeze theorem that

$$\lim_{x\to 0+} \frac{e^{-1/x}}{x^m} = 0.$$

□

Define $p_1(x) = 1$. For each integer $n \geq 2$, we define

$$p_n(x) = x^2 p'_{n-1}(x) - (2nx-1)p_{n-1}(x).$$

Problem 1.7 (3 points)**.** Without computing p_2 explicitly, explain why p_2 is a polynomial of degree 1.

Solution. Since p_1 is a constant, $p'_1 = 0$, and $(2nx-1)p_1(x)$ is a first-degree polynomial. It follows that

$$p_2 = x^2 p'_1(x) - (2nx-1)p_1(x)$$

is a polynomial of degree 1. □

Problem 1.8 (3 points)**.** Suppose that we know that p_n is a polynomial of degree $n-1$. Explain why p_{n+1} is a polynomial of degree n.

Solution. Since p_n is a polynomial of degree $n-1$, the derivative p'_{n-1} is a polynomial of degree $n-2$, and so $x^2 p'_n(x)$ is a polynomial of degree n. Moreover, $(2nx-1)p_n(x)$ is a polynomial of degree n as well, whence

$$p_{n+1}(x) = x^2 p'_n(x) - (2nx-1)p_n(x)$$

is a polynomial of degree n. □

Problem 1.9 (2 points)**.** Conclude from Problem 1.7 and Problem 1.8 that, for each integer $n \geq 1$, the polynomial p_n is of degree $n-1$. What is needed here?

Solution. This follows from Problem 1.7 and Problem 1.8 by mathematical induction. □

Problem 1.10 (4 points)**.** Use Problem 1.6 to show that

$$f'(x) = \begin{cases} \frac{p_1(x)}{x^2} f(x) & \text{if } x > 0; \\ 0 & \text{if } x \leq 0. \end{cases}$$

Solution. If $x > 0$, then

$$f'(x) = e^{-1/x} \cdot \frac{1}{x^2} = \frac{p_1(x)}{x^2} f(x).$$

If $x < 0$, then $f(x) = 0$, and so $f'(x) = 0$ for all $x < 0$. If $x = 0$, then

$$\lim_{h^-\to 0} \frac{f(h) - f(0)}{h} \lim_{h^-\to 0} \frac{0-0}{h} = 0$$

and

$$\lim_{h^+\to 0}\frac{f(h)-f(0)}{h}=\lim_{h^+\to 0}\frac{e^{-1/h}}{h}=0$$

by Problem 1.6. It follows that

$$f'(0)=0.$$

□

Problem 1.11 (4 points)**.** Suppose that, for a fixed integer $n \geq 1$, we know that

$$f^{(n)}(x)=\begin{cases}\frac{p_n(x)}{x^{2n}}f(x) & \text{if } x>0;\\ 0 & \text{if } x\leq 0.\end{cases}$$

Use this fact, Problem 1.6, and Problem 1.10 to show that

$$f^{(n+1)}(x)=\begin{cases}\frac{p_{n+1}(x)}{x^{2(n+1)}}f(x) & \text{if } x>0;\\ 0 & \text{if } x\leq 0.\end{cases}$$

Solution. If $x>0$, then Problem 1.10 implies that

$$\begin{aligned}
f^{(n+1)}(x) &= \frac{d}{dx}\frac{p_n(x)}{x^{2n}}f(x)\\
&= \frac{x^{2n}p_n'(x)-(2n)p_n(x)x^{2n-1}}{x^{4n}}f(x)+\frac{p_n(x)}{x^{2n}}f'(x)\\
&= \frac{x^2p_n'(x)-(2nx)p_n(x)}{x^{2(n+1)}}f(x)+\frac{p_n(x)}{x^{2(n+1)}}f(x)\\
&= \frac{x^2p_n'(x)-(2nx-1)p_n(x)}{x^{2(n+1)}}f(x)\\
&= \frac{p_{n+1}(x)}{x^{2(n+1)}}f(x).
\end{aligned}$$

If $x<0$, then $f^{(n)}(x)=0$, and so

$$f^{(n+1)}(x)=0$$

for all $x<0$.

If $x=0$, then

$$\lim_{h\to 0^-}\frac{f^{(n)}(h)-f^{(n)}(0)}{h}=\lim_{h\to 0^-}\frac{0-0}{h}=0$$

and

$$\begin{aligned}
\lim_{h\to 0^+}\frac{f^{(n)}(h)-f^{(n)}(0)}{h} &= \lim_{h\to 0^-}\frac{\frac{p_n(h)}{h^{2n}}f(h)}{h}=\lim_{h\to 0^-}\frac{p_n(h)f(h)}{h^{2n+1}}\\
&= \lim_{h\to 0^-}p_n(h)\lim_{h\to 0^-}\frac{f(h)}{h^{2n+1}}=\left(\lim_{h\to 0^-}p_n(h)\right)\cdot 0=0
\end{aligned}$$

by Problem 1.6. It follows that

$$f^{(n+1)}(0)=0.$$

□

Problem 1.12 (2 points)**.** Conclude from Problem 1.10 and Problem 1.11 that

$$f^{(n)}(x) = \begin{cases} \frac{p_n(x)}{x^{2n}} f(x) & \text{if } x > 0; \\ 0 & \text{if } x \le 0; \end{cases}$$

for all integers $n \ge 1$. What is needed here?

Solution. This follows from Problem 1.10 and Problem 1.11 by mathematical induction. □

Problem 1.13 (2 points)**.** Use Problem 1.12 to write down the Taylor series of f centered at 0. Does it equal $f(x)$ on any interval of the form $(-R, R)$?

Solution. Since $f^{(n)}(0) = 0$ for all $n \ge 0$, we see that the Taylor series of f centered at 0 is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = 0.$$

Since $f(x) \neq 0$ for all $x > 0$, it follows that the Taylor series of f centered at 0 cannot equal f on any interval of the form $(-R, R)$. □

We have thus shown that there exists an infinitely differentiable function (a *smooth* function) that does not have a valid Taylor series expansion (a non-*analytic* function). As it turns out, functions that have a valid Taylor series expansion everywhere on their domains (*analytic* functions) have extremely rigid structures that are not shared by smooth functions. A systematic study of analytic functions belongs to the realm of *complex analysis*.

2. Fourier Cosine Series (19 points)

We have just seen that some functions do not admit a valid Taylor expansion. The goal of this section is to introduce another method of decomposing a function into simpler pieces other than polynomials (as in the Taylor series case).

Problem 2.1 (2 points)**.** Fix an integer $N \ge 0$, and let $c_0, \ldots, c_N$ be constants. Show that

$$g_N(x) = \sum_{n=0}^{N} c_n \cos(nx)$$

is a 2π-periodic function, i.e.,

$$g_N(x + 2\pi) = g_N(x)$$

for all x.

Hint: use the fact that

$$\cos(y + 2n\pi) = \cos y$$

for each real number y and every integer n.

Solution. Observe that

$$g_N(x + 2\pi) = \sum_{n=0}^{N} c_n \cos(nx + 2\pi n) = \sum_{n=0}^{N} c_n \cos(nx) = g_N(x)$$

for all choices of x. It follows that g_N is 2π-periodic. □

A natural question to ask at this stage is whether it is possible to write *every* 2π-periodic function as a sum of cosines. Taking a cue from Taylor series, let us study series representations of the form

$$f(x) = \sum_{n=0}^{\infty} A_n \cos(nx), \tag{2.1}$$

where A_n are suitably chosen constants.

What should A_n be? We need preliminary results:

Problem 2.2 (2 points)**.** Let m and n be positive integers. Show that

$$\frac{\cos((m+n)x) + \cos((m-n)x)}{2} = \cos(mx)\cos(nx).$$

Solution. By the angle sum formula,

$$\cos((m+n)x) = \cos(mx+nx) = \cos(mx)\cos(nx) - \sin(mx)\sin(nx).$$

Similarly, the angle difference formula implies that

$$\cos((m-n)x) = \cos(mx-nx) = \cos(mx)\cos(nx) + \sin(mx)\sin(nx).$$

It follows that

$$\frac{\cos((m+n)x) + \cos((m-n)x)}{2} = \cos(mx)\cos(nx).$$

□

Problem 2.3 (3 points)**.** Let m and n be positive integers. Use Problem 2.2 to show that

$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,dx = \begin{cases} \pi & \text{if } m = n; \\ 0 & \text{if } m \neq n. \end{cases}$$

Solution. Problem 2.2 implies that

$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,dx = \frac{1}{2}\int_{-\pi}^{\pi} \cos((m+n)x)\,dx + \frac{1}{2}\int_{-\pi}^{\pi} \cos((m-n)x)\,dx.$$

If $m \neq n$, then both integrals are 0. If $m = n$, then

$$\frac{1}{2}\int_{-\pi}^{\pi} \cos((m+n)x) = 0,$$

but

$$\frac{1}{2}\int_{-\pi}^{\pi} \cos((m-n)x)\,dx = \frac{1}{2}\int_{-\pi}^{\pi} \cos 0\,dx = \frac{1}{2}\cdot 2\pi = \pi.$$

□

Now, if we assume that there exist constants A_n such that formula (2.1) holds, then we can apply the result of Problem 2.2 as follows: for each integer $n \geq 1$,

$$\begin{aligned} \int_{-\pi}^{\pi} f(t)\cos(nt)\,dt &= \int_{-\pi}^{\pi} \left(\sum_{m=0}^{\infty} A_m \cos(mt)\right)\cos(nt)\,dt \\ &= \sum_{m=0}^{\infty} A_m \int_{-\pi}^{\pi} \cos(mt)\cos(nt)\,dt \\ &= A_n \pi. \end{aligned}$$

Therefore, it would be sensible to set

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt$$

for each $n \geq 1$. As for the $n = 0$ case, we observe that

$$\int_{-\pi}^{\pi} f(t)\,dt = \int_{-\pi}^{\pi}\sum_{m=0}^{\infty} A_m\cos(mt)\,dt = \sum_{m=0}^{\infty} A_m\int_{-\pi}^{\pi}\cos(mt)\,dt = A_0\cdot 2\pi.$$

Therefore, it would be reasonable to define

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt.$$

Setting

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt,$$

we see that (2.1) becomes:

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} A_n\cos(nx) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt + \sum_{n=1}^{\infty}\left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt\right)\cos(nx) \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nx). \end{aligned}$$

We call

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nx)$$

the *Fourier cosine series* of f. For each integer $n \geq 0$, we call a_n the *nth Fourier cosine coefficient* of f.

Unfortunately, the Fourier cosine series of a function certainly does not equal the function at all times.

Problem 2.4 (3 points)**.** Let $f(x) = x$ on $[-\pi, \pi]$ and compute the Fourier cosine series of f. Conclude that the Fourier cosine series of f does not equal f anywhere on $[-\pi, \pi]$, other than at $x = 0$.

Solution. Since $f(x)$ is odd, and $\cos(nx)$ is even, it follows that $f(x)\cos(nx)$ is odd. We conclude that

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos(nx)\,dx = 0$$

for all $n \geq 0$. The Fourier cosine series of f is thus given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nx) = 0,$$

and this cannot equal $f(x)$ for any $x \neq 0$. □

What extra conditions do we need on the function f? For one, it would be reasonable to demand that f be an even function.

Problem 2.5 (2 points)**.** Fix an integer $N \geq 0$, and let $c_0, \ldots, c_N$ be constants. Show that

$$g_N(x) = \sum_{n=0}^{N} c_n \cos(nx)$$

is an even function, i.e.,

$$g_N(-x) = g_N(x)$$

for all x.

Solution. Observe that

$$g_N(-x) = \sum_{n=0}^{N} c_n \cos(-nx) = \sum_{n=0}^{N} c_n \cos(nx) = g_N(x)$$

for all choices of x. It follows that g_N is even. □

As it turns out, all "nice" even functions have Fourier cosine series that converge to the original functions. The precise determination of proper "niceness" is beyond the scope of this course, but rest assured that most even functions we encounter are "nice" enough.

Let us now consider an example of a function that has a convergent Fourier cosine series.

Problem 2.6 (4 points)**.** Let $f(x) = |x|$ on $[-\pi, \pi]$. (This is one period of the so-called "triangle wave".) Compute the Fourier cosine series of f.

Solution. Since $|x|$ and $\cos(nx)$ are both even, $|x|\cos(nx)$ is even as well. Therefore,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \cos(nx)\, dx = \frac{2}{\pi}\int_0^{\pi} x\cos(nx)\, dx$$

If $n = 0$, then

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\, dx = \pi.$$

If $n \geq 1$, then we integrate by substitution and then integrate by parts to obtain the following:

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi} x\cos(nx)\, dx \\ &= \frac{2}{\pi n^2}\int_0^{n\pi} t\cos t\, dt \\ &= \frac{2}{\pi n^2}\left((n\pi)\sin(n\pi) - 0\cdot\sin(0) - \int_0^{n\pi}\sin t\, dt\right) \\ &= -\frac{2}{\pi n^2}\int_0^{n\pi}\sin t\, dt. \end{aligned}$$

Since

$$\int_0^{n\pi}\sin t\, dt = \begin{cases} 2 & \text{if } n \text{ is odd;} \\ 0 & \text{if } n \text{ is even;} \end{cases}$$

we see that

$$a_n = \begin{cases} -\frac{4}{\pi n^2} & \text{if } n \text{ is odd;} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

It follows that the Fourier cosine series of f is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) = \frac{\pi}{2} + \sum_{n=1}^{\infty} -\frac{4}{\pi(2n-1)^2} \cos(2n-1)x.$$

□

Here is a nifty application:

Problem 2.7 (3 points)**.** We continue to let $f(x) = |x|$ on $[-\pi, \pi]$. Since

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx),$$

we see that

$$0 = f(0) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n.$$

Use Problem 2.6 to show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Solution. The previous problem implies that

$$|x| = \frac{\pi}{2} + \sum_{n=1}^{\infty} -\frac{4}{\pi(2n-1)^2} \cos(2n-1)x.$$

Plugging in $x = 0$, we obtain the following identity:

$$0 = \frac{\pi}{2} + \sum_{n=1}^{\infty} -\frac{4}{\pi(2n-1)^2}.$$

Rearranging, we obtain

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

Now, we observe that

$$\sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Since

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{(2n)^2} + \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2},$$

it follows that

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{3}{4} \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

We conclude that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{4}{3} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{6},$$

as was to be shown. □

Fourier cosine series, along with its counterpart Fourier sine series, is an extremely powerful tool that many disciplines—such as engineering, physics, biology, chemistry, computer science, and finance—call upon on a regular basis. Fourier series is the main object of study in the field of *Fourier analysis*.