CALCULUS II, SUMMER 2015 - EXTRA PROBLEM SET

`50 extra credit points`

`Name:` ____ `Score:` ______/ 50

Use this worksheet as the cover sheet for your write-up: write your name on this page, and staple this sheet to the front of your homework packet.

Please indicate clearly which problems you have worked on. You will receive no credit for submitting solutions that the grader cannot read and understand—be sure to write legibly!

1. A Smooth, Non-analytic Function (31 points)

Throughout this problem set, we consider the piecewise-defined function

$$f(x) = \begin{cases} e^{-1/x} & \text{if } x > 0; \\ 0 & \text{if } x \leq 0. \end{cases}$$

Problem 1.1 (2 points)**.** Show that

$$\left(\frac{1}{x}\right)^{m+1} \leq \sum_{n=0}^{\infty} \frac{(m+1)!}{n!} \left(\frac{1}{x}\right)^n$$

for each integer $m \geq 0$ and every real number $x > 0$.

Problem 1.2 (1 point)**.** Use Problem 1.1 to show that

$$\frac{1}{x^m} \leq x \sum_{n=0}^{\infty} \frac{(m+1)!}{n!} \left(\frac{1}{x}\right)^n$$

for each integer $m \geq 0$ and every real number $x > 0$.

Problem 1.3 (2 points)**.** Use the Taylor series expansion of the exponential function e^x centered at 0 to show that

$$\sum_{n=0}^{\infty} \frac{(m+1)!}{n!} \left(\frac{1}{x}\right)^n = (m+1)! e^{1/x}$$

for each integer $m \geq 0$ and every real number $x > 0$.

Problem 1.4 (2 points)**.** Use Problem 1.2 and Problem 1.3 to show that

$$\frac{1}{x^m} \leq x(m+1)! e^{1/x}$$

for each integer $m \geq 0$ and every real number $x > 0$.

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Problem 1.5 (3 points)**.** Use Problem 1.4 to show that

$$0 \leq \frac{e^{-1/x}}{x^m} \leq x(m+1)!$$

for each integer $m \geq 0$ and every real number $x > 0$.

Problem 1.6 (1 point)**.** Use Problem 1.5 to show that

$$\lim_{x \to 0^+} \frac{e^{-1/x}}{x^m} = 0$$

for each integer $m \geq 0$.

Define $p_1(x) = 1$. For each integer $n \geq 2$, we define

$$p_n(x) = x^2 p'_{n-1}(x) - (2nx - 1)p_{n-1}(x).$$

Problem 1.7 (3 points)**.** Without computing p_2 explicitly, explain why p_2 is a polynomial of degree 1.

Problem 1.8 (3 points)**.** Suppose that we know that p_n is a polynomial of degree $n-1$. Explain why p_{n+1} is a polynomial of degree n.

Problem 1.9 (2 points)**.** Conclude from Problem 1.7 and Problem 1.8 that, for each integer $n \geq 1$, the polynomial p_n is of degree $n-1$. What is needed here?

Problem 1.10 (4 points)**.** Use Problem 1.6 to show that

$$f'(x) = \begin{cases} \frac{p_1(x)}{x^2} f(x) & \text{if } x > 0; \\ 0 & \text{if } x \leq 0. \end{cases}$$

Problem 1.11 (4 points)**.** Suppose that, for a fixed integer $n \geq 1$, we know that

$$f^{(n)}(x) = \begin{cases} \frac{p_n(x)}{x^{2n}} f(x) & \text{if } x > 0; \\ 0 & \text{if } x \leq 0. \end{cases}$$

Use this fact, Problem 1.6, and Problem 1.10 to show that

$$f^{(n+1)}(x) = \begin{cases} \frac{p_{n+1}(x)}{x^{2(n+1)}} f(x) & \text{if } x > 0; \\ 0 & \text{if } x \leq 0. \end{cases}$$

Problem 1.12 (2 points)**.** Conclude from Problem 1.10 and Problem 1.11 that

$$f^{(n)}(x) = \begin{cases} \frac{p_n(x)}{x^{2n}} f(x) & \text{if } x > 0; \\ 0 & \text{if } x \leq 0; \end{cases}$$

for all integers $n \geq 1$. What is needed here?

Problem 1.13 (2 points)**.** Use Problem 1.12 to write down the Taylor series of f centered at 0. Does it equal $f(x)$ on any interval of the form $(-R, R)$?

We have thus shown that there exists an infinitely differentiable function (a *smooth* function) that does not have a valid Taylor series expansion (a non-*analytic* function). As it turns out, functions that have a valid Taylor series expansion everywhere on their domains (*analytic* functions) have extremely rigid structures that are not shared by smooth functions. A systematic study of analytic functions belongs to the realm of *complex analysis*.

2. Fourier Cosine Series (19 points)

We have just seen that some functions do not admit a valid Taylor expansion. The goal of this section is to introduce another method of decomposing a function into simpler pieces other than polynomials (as in the Taylor series case).

Problem 2.1 (2 points)**.** Fix an integer $N \geq 0$, and let $c_0, \ldots, c_N$ be constants. Show that

$$g_N(x) = \sum_{n=0}^{N} c_n \cos(nx)$$

is a 2π-periodic function, i.e.,

$$g_N(x + 2\pi) = g_N(x)$$

for all x.

Hint: use the fact that

$$\cos(y + 2n\pi) = \cos y$$

for each real number y and every integer n.

A natural question to ask at this stage is whether it is possible to write *every* 2π-periodic function as a sum of cosines. Taking a cue from Taylor series, let us study series representations of the form

$$f(x) = \sum_{n=0}^{\infty} A_n \cos(nx), \tag{2.1}$$

where A_n are suitably chosen constants.

What should A_n be? We need preliminary results:

Problem 2.2 (2 points)**.** Let m and n be positive integers. Show that

$$\frac{\cos((m+n)x) + \cos((m-n)x)}{2} = \cos(mx)\cos(nx).$$

Problem 2.3 (3 points)**.** Let m and n be positive integers. Use Problem 2.2 to show that

$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,dx = \begin{cases} \pi & \text{if } m = n; \\ 0 & \text{if } m \neq n. \end{cases}$$

Now, if we assume that there exist constants A_n such that formula (2.1) holds, then we can apply the result of Problem 2.2 as follows: for each integer $n \geq 1$,

$$\begin{aligned} \int_{-\pi}^{\pi} f(t)\cos(nt)\,dt &= \int_{-\pi}^{\pi} \left(\sum_{m=0}^{\infty} A_m \cos(mt) \right) \cos(nt)\,dt \\ &= \sum_{m=0}^{\infty} A_m \int_{-\pi}^{\pi} \cos(mt)\cos(nt)\,dt \\ &= A_n \pi. \end{aligned}$$

Therefore, it would be sensible to set

$$A_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\cos(nt)\,dt$$

for each $n \geq 1$. As for the $n = 0$ case, we observe that

$$\int_{-\pi}^{\pi} f(t)\,dt = \int_{-\pi}^{\pi} \sum_{m=0}^{\infty} A_m \cos(mt)\,dt = \sum_{m=0}^{\infty} A_m \int_{-\pi}^{\pi} \cos(mt)\,dt = A_0 \cdot 2\pi.$$

Therefore, it would be reasonable to define

$$A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\,dt.$$

Setting

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\cos(nt)\,dt,$$

we see that (2.1) becomes:

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} A_n \cos(nx) \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\,dt + \sum_{n=1}^{\infty} \left(\frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\cos(nt)\,dt \right) \cos(nx) \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx). \end{aligned}$$

We call

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$$

the *Fourier cosine series* of f. For each integer $n \geq 0$, we call a_n the *nth Fourier cosine coefficient* of f.

Unfortunately, the Fourier cosine series of a function certainly does not equal the function at all times.

Problem 2.4 (3 points)**.** Let $f(x) = x$ on $[-\pi, \pi]$ and compute the Fourier cosine series of f. Conclude that the Fourier cosine series of f does not equal f anywhere on $[-\pi, \pi]$, other than at $x = 0$.

What extra conditions do we need on the function f? For one, it would be reasonable to demand that f be an even function.

Problem 2.5 (2 points)**.** Fix an integer $N \geq 0$, and let $c_0, \ldots, c_N$ be constants. Show that

$$g_N(x) = \sum_{n=0}^{N} c_n \cos(nx)$$

is an even function, i.e.,

$$g_N(-x) = g_N(x)$$

for all x.

As it turns out, all "nice" even functions have Fourier cosine series that converge to the original functions. The precise determination of proper "niceness" is beyond the scope of this course, but rest assured that most even functions we encounter are "nice" enough.

Let us now consider an example of a function that has a convergent Fourier cosine series.

Problem 2.6 (4 points)**.** Let $f(x) = |x|$ on $[-\pi, \pi]$. (This is one period of the so-called "triangle wave".) Compute the Fourier cosine series of f.

Here is a nifty application:

Problem 2.7 (3 points)**.** We continue to let $f(x) = |x|$ on $[-\pi, \pi]$. Since

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx),$$

we see that

$$0 = f(0) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n.$$

Use Problem 2.6 to show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Fourier cosine series, along with its counterpart Fourier sine series, is an extremely powerful tool that many disciplines—such as engineering, physics, biology, chemistry, computer science, and finance—call upon on a regular basis. Fourier series is the main object of study in the field of *Fourier analysis*.