CALCULUS II, SUMMER 2015 - SET THEORY

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Set theory is not covered in the course, but we will make use of several set-theoretic notions throughout the course. This set of notes is meant to serve as a handy reference for all such notions.

1. What is a set?

A *set*, in mathematics, is a collection of objects with clear membership. That is, an object must either be in the set, or not in the set; "sort of in the set" is not acceptable. An object in a set is called an *element* of the set.

Given a set X, we write $x \in X$ to denote that x is an element of the set X. In contrast, $x \notin X$ means that x is not an element of the set X.

Example 1.1. All students in the NYU CAS class of 2019.

Example 1.1 is a set, as we can tell clearly who (or what) is in the "NYU CAS class of 2019." Richard B. Freshman is in the set; Jeremiah the bullfrog is not.

Example 1.2. All female students in the NYU CAS class of 2019.

Example 1.2 is also a set. We note that Example 1.2 describes a set smaller than that described by Example 1.1. For one, Richard B. Freshman is not in the second set. Everyone in the second set, however, belongs to the first set.

Example 1.2 is an example of a *subset*. The second set is a subset of the first set.

Example 1.3. All tall students in the NYU CAS class of 2019.

Example 1.3 is **not** a set. Though we may argue that we *know* who is tall or not, the concept of "tall" is just too vague.

2. Set notations

There are two ways of writing down a set: *roster* notation, and *set-builder* notation.

To use roster notation, we simply write down all elements of the set, and enclose them with curly brackets. The result is a set, as we have indicated explicitly what belongs to the set.

Example 2.1. {Brooklyn, Manhattan, Queens, Staten Island, The Bronx}

Example 2.1 is a set, written in roster notation. In fact, it is the set of all five boroughs of New York City.

Example 2.2. {The Bronx, Brooklyn, Queens, Manhattan, Staten Island}

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Example 2.2 is also the set of all five boroughs of New York City. It would be counterintuitive to say that Example 2.1 and Example 2.2 denote two different sets, as both are, well, the set of all five boroughs of New York City.

Indeed, in roster notation, *the order of the elements does not matter.*

Example 2.3. {Brooklyn, Brooklyn, Brooklyn, Brooklyn, Manhattan, Manhattan, Manhattan, Queens, Queens, Staten Island, The Bronx}

Likewise, Example 2.3 also denotes the set of all five boroughs of New York City. Therefore, in roster notation, *duplicates are ignored*.

Roster notation is clean and simple, but it isn't all that easy to use to describe large sets. For one, writing Example 1.1 in roster notation would result in a very, very long list of names. (Indeed, a *roster* of all freshmen!)

this is where set-builder notation comes in. Instead of writing down every single element of a set, set-builder notation specifies a condition from which the set can be built.

Example 2.4. $\{x$ is a person : x is a student in the NYU CAS class of 2019$\}$

Example 2.4 is an example of a set, written in set-builder notation. The above notation translates to the following: "a set of all people x such that x is a student in the NYU CAS class of 2019." The first part specifies what kind of elements we wish to collect in a set; the second part specifies the set.

We remark that a change of variables does not change the set:

Example 2.5. {☺ is a person : ☺ is a student in the NYU CAS class of 2019}

: is a shorthand notation for "such that", and | is often used in place of : as well:

Example 2.6. $\{x$ is a person | x is a student in the NYU CAS class of 2019$\}$

Note that logical connectives such as *and* or *or* can be used in the second part of set-builder notation.

Example 2.7. $\{x$ is a person | x is a student *and* x is in the NYU CAS class of 2019$\}$

Example 2.7 denotes the same set as Example 1.1, Example 2.4, Example 2.5, and Example 2.6.

3. Sets of numbers, Part 1

Let us now introduce the frequently-used sets of numbers in mathematics.

- $\mathbb{N}$ is the set of *natural numbers*:
$$1, 2, 3, 4, 5, \ldots.$$
- $\mathbb{Z}$ is the set of *integers*:
$$\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots.$$
- $\mathbb{Q}$ is the set of *rational numbers*, which are numbers that can be written as a ratio $\frac{a}{b}$ of an integer a and a non-zero integer b.
- $\mathbb{R}$ is the set of all *real numbers*, which are the numbers that can be written as a decimal, possibly an infinite one. For example,
$$\sqrt{2} = 1.414213562\ldots$$

and

$$\pi = 3.141592\ldots.$$

Consider the following examples of sets of numbers:

Example 3.1. $\{1, 2, 3, 4, 5\}$

Example 3.2. $\{x \in \mathbb{Z} : 1 \leq x \leq 5\}$

Example 3.3. $\{x \in \mathbb{R} : 1 \leq x \leq 5\}$

Example 3.4. $\{x \in \mathbb{R} : 1 \leq x \leq 3 \text{ or } 3 < x \leq 5\}$

Examples 3.1 and 3.2 denote the same set. Examples 3.3 and 3.4 denote the same set (can you see why?). Examples 3.2 and 3.4 do **not** denote the same set: 1.5 and $\sqrt{2}$ are elements of Example 3.3 but are not elements of Example 3.2.

In calculus courses, "$\in \mathbb{R}$" in the first part of set-builder notation is often omitted, as *all* numbers we deal with are assumed to be real numbers. Therefore, the following denotes the same set as Examples 3.3 and 3.4:

Example 3.5. $\{x : 1 \leq x \leq 5\}$.

4. Set Operations

Let X and Y be sets. How do we describe a set whose element consists of elements of X *and* elements of Y? We could use set-builder notation, to be sure:

$$\{x : x \in X \text{ and } x \in Y\}.$$

The resulting set is called the *intersection* of X and Y and is denoted by $X \cap Y$. Similarly,

$$\{x : x \in X \text{ or } x \in Y\}$$

is called the *union* of X and Y and is denoted by $X \cup Y$.

Let us consider some examples of unions and intersections of sets.

Example 4.1. $\{1, 2, 3\} \cup \{4, 5, 6\} = \{1, 2, 3, 4, 5, 6\}$.

Example 4.2. $\{1, 2, 3, 4\} \cup \{3, 4, 5, 6\} = \{1, 2, 3, 4, 5, 6\}$, because $\{1, 2, 3, 4, 5, 6\} = \{1, 2, 3, 4, 3, 4, 5, 6\}$.

Example 4.3. $\{2, 4, 6\} = \{1, 2, 3, 6\} = \{2, 6\}$.

But, of course, two sets many not share any element. The resulting set in this case is called the *empty set*, which we denote by $\varnothing$.

Example 4.4. $\{1, 3, 5\} \cap \{2, 4, 6\} = \varnothing$.

The empty set has no element. Therefore, if we take the union of a set and the empty set, the resulting set is the original one, unchanged:

Example 4.5. $\{4, 6, 8, 10\} \cup \varnothing = \{4, 6, 8, 10\}$.

On the other hand, if we take the intersection of a set with the empty set, we obtain the empty set—because no set can share any element with the set with no elements.

Example 4.6. $\{4, 6, 8, 10\} \cap \varnothing = \varnothing$.

5. Sets of Numbers, Part 2: Intervals

An *interval*, intuitively, is a segment of numbers:

Example 5.1. All real numbers between 1 and 4.

Using set-roster notation, we can write the above as follows:

Example 5.2. $\{x : 1 < x < 4\}$.

Because intervals appear frequently in calculus, we introduce specialized notations for them.

Interval Notation	Set Notation
$[a, b]$	$\{x : a \leq x \leq b\}$
(a, b)	$\{x : a < x < b\}$
$[a, b)$	$\{x : a \leq x < b\}$
$(a, b]$	$\{x : a < x \leq b\}$

Here a and b are real numbers such that $a < b$.

Example 5.1 can then be written as follows:

Example 5.3. $(1, 4)$

The notion of intervals can be extended to include unbounded sets of numbers as well.

Example 5.4. All real numbers larger than 4.

To use the interval notation to describe the above set, we use the positive infinity symbol ∞:

Example 5.5. $(4, \infty)$

Similarly, the negative infinity symbol $-\infty$ is used as well:

Example 5.6. $(-\infty, 3) = \{x : x < 3\}$.

We remark that square brackets [and] can never be used for ∞ or $-\infty$, because infinities are not numbers and thus intervals of real numbers cannot have them as elements.

Since intervals are sets, the set operations can be applied to any two intervals. Sometimes, the net result of such an operation is another interval, as shown below:

Example 5.7. $[1, 4] \cup [4, 7] = [1, 7]$

Example 5.8. $(1, 2) \cup (1.5, 8] = (1, 8]$

Example 5.9. $[-2, 2] \cap (-1, 3) = (-1, 2]$

Example 5.10. $[-2, 2] \cap (-1, 3) = (-1, 2]$

Example 5.11. $(-\infty, \sqrt{2}) \cap (-4.5, \infty) = (-4.5, \sqrt{2})$

Some operations, however, do not produce a single interval:

Example 5.12. $(3, 6) \cup (-1, 2)$

Moreover, taking the intersection of two intervals could very well produce the empty set:

Example 5.13. $(1, 2) \cap (2, 3) = \varnothing$.