CALCULUS II, SUMMER 2015 - TELESCOPING SUMS

A *telescoping sum* is an infinite sum of the type

$$\sum_{n=1}^{\infty} a_n - a_{n+k}$$

for some positive integer k.

Example 1. Let us compute

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}.$$

We first note that

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1},$$

so that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1}.$$

For each $N \geq 1$, we let

$$S_N = \sum_{n=1}^{N} \frac{1}{n} - \frac{1}{n+1}.$$

Observe that

$$S_N = \sum_{n=1}^{N} \frac{1}{n} - \sum_{n=1}^{N} \frac{1}{n+1} = \sum_{n=1}^{N} \frac{1}{n} - \sum_{n=2}^{N+1} \frac{1}{n}$$
$$= 1 + \left(\sum_{n=2}^{N} \frac{1}{n} - \sum_{n=2}^{N} \frac{1}{n} \right) - \frac{1}{N+1} = 1 - \frac{1}{N+1}.$$

It now follows that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{N \to \infty} S_N = \lim_{N \to \infty} 1 - \frac{1}{N+1} = 1.$$

□

We might be tempted to write

(1) $$\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1} = \sum_{n=1}^{\infty} \frac{1}{n} - \sum_{n=1}^{\infty} \frac{1}{n+1}$$

and then reindex the second series to obtain

$$\sum_{n=1}^{\infty} \frac{1}{n} - \sum_{n=1}^{\infty} \frac{1}{n+1} = \sum_{n=1}^{\infty} \frac{1}{n} - \sum_{n=2}^{\infty} \frac{1}{n} = 1 + \sum_{n=2}^{\infty} \frac{1}{n} - \sum_{n=2}^{\infty} \frac{1}{n} = 1.$$

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While we obtain the same answer this way, it is important to note that (1) is an illegitimate operation. Indeed,

$$\sum_{n=1}^{\infty} a_n - b_n = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$$

is valid (by the algebraic limit theorem) only if both $\sum a_n$ and $\sum b_n$ are convergent series. In our case, neither $\sum \frac{1}{n}$ nor $\sum \frac{1}{n+1}$ is a convergent series, and so we cannot apply the algebraic limit theorem.

The next example illustrates that our luck runs out rather quickly if we fail to pay attention to this subtlety.

Example 2. We compute

$$\sum_{n=1}^{\infty} \cos\frac{\pi}{n} - \cos\frac{\pi}{n+1}.$$

For each $N \geq 1$, we define

$$S_N = \sum_{n=1}^{N} \cos\frac{\pi}{n} - \cos\frac{\pi}{n+1}$$

and observe that

$$\begin{aligned}\sum_{n=1}^{N} \cos\frac{\pi}{n} - \cos\frac{\pi}{n+1} = \sum_{n=1}^{N} \cos\frac{\pi}{n} - \sum_{n=1}^{N} \cos\frac{\pi}{n+1} &= \sum_{n=1}^{N} \cos\frac{\pi}{n} - \sum_{n=2}^{N+1} \cos\frac{\pi}{n} \\ &= \cos\pi + \sum_{n=2}^{N} \cos\frac{\pi}{n} - \sum_{n=2}^{N} \cos\frac{\pi}{n} - \cos\frac{\pi}{N+1} \\ &= \cos\pi - \cos\frac{\pi}{N+1} = -1 - \cos\frac{\pi}{N+1}\end{aligned}$$

It follows that

$$\sum_{n=1}^{\infty} \cos\frac{\pi}{n} - \cos\frac{\pi}{n+1} = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \cos 1 - \cos\frac{\pi}{N+1} = -1 - 1 = -2.$$

□

If we try to manipulate the infinite sums directly, we get

$$\begin{aligned}\sum_{n=1}^{\infty} \cos\frac{\pi}{n} - \cos\frac{\pi}{n+1} = \sum_{n=1}^{\infty} \cos\frac{\pi}{n} - \sum_{n=1}^{\infty} \cos\frac{\pi}{n+1} &= \sum_{n=1}^{\infty} \cos\frac{\pi}{n} - \sum_{n=2}^{\infty} \cos\frac{\pi}{n} \\ &= \cos\frac{\pi}{1} + \sum_{n=2}^{\infty} \cos\frac{\pi}{n} - \sum_{n=2}^{\infty} \cos\frac{\pi}{n} \\ &= -1,\end{aligned}$$

which is wrong.

Example 3. We use the telescoping sum technique to produce an upper bound of the finite sum

$$\sum_{n=1}^{N} \sin n$$

that is independent of the choice of N. To this end, we recall the trigonometric identity

$$2\sin A \sin B = \cos(A-B) - \cos(A+B).$$

Using this identity, we see that

$$\sum_{n=1}^{N} \sin N = \frac{1}{2\sin\frac{1}{2}} \sum_{n=1}^{N} 2\sin\frac{1}{2}\sin n = \frac{1}{2\sin\frac{1}{2}} \sum_{n=1}^{N} \cos\left(n-\frac{1}{2}\right) - \cos\left(n+\frac{1}{2}\right).$$

Observe that

$$\begin{aligned}
&\sum_{n=1}^{N} \cos\left(n-\frac{1}{2}\right) - \cos\left(n+\frac{1}{2}\right) \\
&= \sum_{n=1}^{N} \cos\left(n-\frac{1}{2}\right) - \sum_{n=1}^{N} \cos\left(n+\frac{1}{2}\right) \\
&= \sum_{n=1}^{N} \cos\left(n-\frac{1}{2}\right) - \sum_{n=2}^{N+1} \cos\left(n-\frac{1}{2}\right) \\
&= \cos\frac{1}{2} + \sum_{n=2}^{N} \cos\left(n-\frac{1}{2}\right) - \sum_{n=2}^{N} \cos\left(n-\frac{1}{2}\right) - \cos\frac{2N+1}{2} \\
&= \cos\frac{1}{2} - \cos\frac{2N+1}{2}.
\end{aligned}$$

It now follows from the triangle inequality that

$$\left|\sum_{n=1}^{N} \sin n\right| = \frac{|\cos\frac{1}{2} + \cos\frac{2N+1}{2}|}{|2\sin\frac{1}{2}|} \leq \frac{|\cos\frac{1}{2}| + |\cos\frac{2N+1}{2}|}{2\sin\frac{1}{2}} \leq \frac{1+1}{2\sin\frac{1}{2}} = \frac{1}{\sin\frac{1}{2}}.$$

Here we have made use of the fact that $|\cos x| \leq 1$ for all x. $\square$