CALCULUS II, SUMMER 2015 - WEEKEND PROBLEM SET 1 SOLUTIONS

The evaluation of any indefinite integral must include the constant of integration $+C$; please mark the answer wrong otherwise.

1. The Logarithm (34 points)

Problem 1.1 (10 points)**.** For each $x > 0$, the natural logarithm of x is defined to be the integral

$$\ln x = \int_1^x \frac{1}{t}\, dt.$$

(1) Explain why $\ln 1 = 0$.
(2) Explain why $\frac{d}{dx} \ln x = \frac{1}{x}$ for all $x > 0$.
(3) Explain why $\frac{d}{dx} \ln |x| = \frac{1}{x}$ for all $x \neq 0$. *Hint*: Apply the chain rule, and consider the $x > 0$ case and the $x < 0$ case separately. Conclude that $\int \frac{1}{x}\, dx = \ln |x| + C$.
(4) Explain why $\ln ab = \ln a + \ln b$ for all $a > 0$ and $b > 0$. *Hint*: $\int_1^{ab} \frac{1}{t}\, dt = \int_1^a \frac{1}{t}\, dt + \int_a^{ab} \frac{1}{t}\, dt$.
(5) Explain why $\ln x^r = r \ln x$ for all $r > 0$ and $x > 0$.
Hint: Define $f(x) = \ln x^r - r \ln x$. differentiate f using (2), and apply (1) to show that $f(x) = 0$ for all x. Conclude that $\ln x^r = r \ln x$.

Solution for (1). By the second fundamental theorem of calculus,

$$\ln x = \int_1^x \frac{1}{t}\, dt = F(x) - F(1),$$

where F is an antiderivative of $\frac{1}{t}$. It now follows that

$$\ln 1 = F(1) - F(1) = 0.$$

□

Solution for (2). By the first fundamental theorem of calculus,

$$\frac{d}{dx} \ln x = \frac{d}{dx} \int_1^x \frac{1}{t}\, dt = \frac{1}{x}.$$

□

Solution for (3). By the chain rule,

$$\frac{d}{dx} \ln |x| = \frac{1}{|x|} \frac{d}{dx} |x|.$$

If $x > 0$, then $|x| = x$ and $\frac{d}{dx}|x| = 1$, and so

$$\frac{d}{dx} \ln |x| = \frac{1}{|x|} \frac{d}{dx} |x| = \frac{1}{x}.$$

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If $x < 0$, then $|x| = -x$ and $\frac{d}{dx}|x| = -1$, and so

$$\frac{d}{dx}\ln|x| = \frac{1}{|x|}\frac{d}{dx}|x| = \frac{1}{-x}\cdot(-1) = \frac{1}{x}.$$

□

Solution for (4). Observe that

$$\ln ab = \int_1^{ab}\frac{1}{t}\,dt = \int_1^{a}\frac{1}{t}\,dt + \int_a^{ab}\frac{1}{t}\,dt = \ln a + \int_a^{ab}\frac{1}{t}\,dt.$$

Now, we make the substitution $u = a^{-1}t$, so that $du = a^{-1}\,dt$. We then see that

$$\int_a^{ab}\frac{1}{t}\,dt = \int_{a^{-1}(a)}^{a^{-1}(ab)}\frac{1}{au}a\,du = \int_1^{b}\frac{1}{u}\,du = \ln b.$$

It follows that

$$\ln ab = \ln a + \int_a^{ab}\frac{1}{t}\,dt = \ln a + \ln b,$$

as was to be shown. □

Solution for (5). Consider the function $f(x) = \ln x^r - r\ln x$. Differentiating f, we obtain the following identity:

$$f'(x) = \frac{rx^{r-1}}{x^r} - \frac{r}{x} = \frac{r}{x} - \frac{r}{x} = 0.$$

Since the derivative of f is 0 for all x, it follows that f is constant. Now,

$$f(1) = \ln 1 - r\ln 1 = 0 - 0 = 0,$$

and so $f(x)$ must equal 0 for all x. It follows that

$$\ln x^r = r\ln x.$$

□

Problem 1.2 (2 points)**.** Let f be a differentiable function with a continuous derivative f'. What is $\int \frac{f'(x)}{f(x)}\,dx$?

Solution. We use the substitution $u = f(x)$, so that $du = f'(x)\,dx$. We then see that

$$\int \frac{f'(x)}{f(x)}\,dx = \int \frac{1}{u}\,du = \ln|u| + C = \ln|f(x)| + C.$$

□

Problem 1.3 (4 points)**.** Integrate $\int \tan x\,dx$ and $\int \cot x\,dx$.

Solution. Observe that

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx.$$

We use the substitution $u = \cos x$, so that $du = -\sin x\,dx$. We then see that

$$\int \frac{\sin x}{\cos x}\,dx = -\int \frac{1}{u}\,du = -\ln|u| + C = -\ln|\cos x| + C = \ln|\sec x| + C.$$

As for the second integral, we observe that

$$\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx.$$

We use the substitution $u = \sin x$, so that $du = \cos x\, dx$. We then see that

$$\int \frac{\cos x}{\sin x}\, dx = \int \frac{1}{u}\, du = \ln |u| + C = \ln |\sin x| + C.$$

□

Problem 1.4 (6 points)**.** Integrate $\int \sec x\, dx$ and $\int \csc x\, dx$.

Solution. Observe that

$$\int \sec x\, dx = \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x}\, dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\, dx.$$

We use the substitution $u = \sec x + \tan x$, so that $du = \sec x \tan x + \sec^2 x\, dx$. We then see that

$$\int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\, dx = \int \frac{1}{u}\, du = \ln |u| + C = \ln |\sec x + \tan x| + C.$$

We now observe that

$$\int \csc x\, dx = \int \csc x \frac{\csc x + \cot x}{\csc x + \cot x}\, dx = \int \frac{\csc^2 x + \csc x \cot x}{\csc x + \cot x}\, dx.$$

We use the substitution $u = \csc x + \cot x$, so that $du = -(\csc x \cot x + \cot^2 x)\, dx$. We then see that

$$\int \frac{\csc^2 x + \csc x \cot x}{\csc x + \cot x}\, dx = -\int \frac{1}{u}\, du = -\ln |u| + C = -\ln |\csc x + \cot x| + C.$$

□

Problem 1.5 (6 points)**.** Integrate $\int \sin(\ln x)\, dx$ and $\int \cos(\ln x)\, dx$.

Solution. We integrate $\int \sin(\ln x)\, dx$ by parts. To this end, we set $u = \sin(\ln x)$ and $dv = 1\, dx$, so that $du = x^{-1} \cos(\ln x)\, dx$ and $v = x$. It follows that

$$\int \sin(\ln x)\, dx = uv - \int v\, du = x \sin(\ln x) - \int \cos(\ln x)\, dx. \tag{1}$$

We now integrate $\int \cos(\ln x)\, dx$ by parts. To this end, we set $u = \cos(\ln x)$ and $dv = 1\, dx$, so that $du = -x^{-1} \sin(\ln x)\, dx$ and $v = x$. It follows that

$$\int \cos(\ln x)\, dx = uv - \int v\, du = x \cos(\ln x) + \int \sin(\ln x)\, dx. \tag{2}$$

Substituting (2) into (1), we obtain the following identity:

$$\int \sin(\ln x)\, dx = x \sin(\ln x) - x \cos(\ln x) - \int \sin(\ln x)\, dx. \tag{3}$$

Adding $\int \sin(\ln x)\, dx$ to both sides and dividing through by 2, we see that

$$\int \sin(\ln x)\, dx = \frac{1}{2} x \left[\sin(\ln x) - \cos(\ln x)\right]. \tag{4}$$

Now, we can also substitute (1) into (1) to obtain the following identity:

$$\int \cos(\ln x)\, dx = x \cos(\ln x) + x \sin(\ln x) - \int \cos(\ln x)\, dx. \tag{5}$$

Adding $\int \cos(\ln x)\, dx$ to both sides and dividing through by 2, we see that

$$\int \cos(\ln x)\, dx = \frac{1}{2} x \left[\sin(\ln x) + \cos(\ln x)\right]. \tag{6}$$

□

Problem 1.6 (6 points)**.** Given two integers $m \geq 1$ and $n \geq 1$, derive the reduction formula

$$\int x^m \ln^n x\,dx = \frac{x^{m+1}\ln^n x}{m+1} - \frac{n}{m+1}\int x^m \ln^{n-1} x\,dx.$$

Use the formula to integrate $\int x^{10}\ln^3 x\,dx$.

Solution. We integrate by parts. To this end, we set

$$u = \ln^n x \quad \text{and} \quad dv = x^m\,dx, \tag{7}$$

so that $du = nx^{-1}\ln^{n-1} x\,dx$ and $v = \frac{1}{m+1}x^{m+1}$. We see that

$$\int x^m \ln^n x\,dx = uv - \int vdu = \frac{x^{m+1}\ln^n x}{m+1} - \frac{n}{m+1}\int x^m \ln^{n-1} x\,dx, \tag{8}$$

which is the desired formula.

We now apply the formula repeatedly:

$$\begin{aligned}
\int x^{10}\ln^3 x\,dx &= \frac{x^{11}\ln^3 x}{11} - \frac{3}{11}\int x^{11}\ln^2 x\,dx \\
&= \frac{x^{11}\ln^3 x}{11} - \frac{3}{11}\left(\frac{x^{11}\ln^2 x}{11} - \frac{2}{11}\int x^{10}\ln x\,dx\right) \\
&= \frac{x^{11}\ln^3 x}{11} - \frac{3x^{11}\ln^2 x}{121} + \frac{6}{121}\int x^{10}\ln x\,dx \\
&= \frac{x^{11}\ln^3 x}{11} - \frac{3x^{11}\ln^2 x}{121} + \frac{6}{121}\left(\frac{x^{11}\ln x}{11} - \frac{1}{11}\int x^{10}\,dx\right) \\
&= \frac{x^{11}\ln^3 x}{11} - \frac{3x^{11}\ln^2 x}{121} + \frac{6}{121}\left(\frac{x^{11}\ln x}{11} - \frac{1}{11}\cdot\frac{1}{11}x^{11}\right) + C \\
&= \frac{x^{11}\ln^3 x}{11} - \frac{3x^{11}\ln^2 x}{121} + \frac{6}{121}\left(\frac{x^{11}\ln x}{11} - \frac{1}{121}x^{11}\right) + C \\
&= \frac{x^{11}\ln^3 x}{11} - \frac{3x^{11}\ln^2 x}{121} + \frac{6x^{11}\ln x}{1331} - \frac{6}{14641}x^{11} + C.
\end{aligned}$$

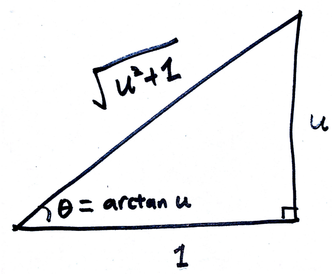
□

2. Tangent half-angle substitution (16 points)

The substitution technique we study in this section is what Michael Spivak, a legendary expositor of mathematics, calls the "world's sneakiest substitution."

If we set $u = \tan\frac{x}{2}$, then $x = 2\arctan u$, so that $dx = \frac{2}{1+u^2}\,du$. The substitution is used to transform the integral of a function obtained by combining sines and cosines via addition, multiplication, and division (a "rational function of sines and cosines") into the integral of a rational function. The latter integral can then be computed by the method of partial fraction decomposition, integration by parts, and integration by substitution.

Problem 2.1 (3 points)**.** What is $\sin(\arctan u)$? What is $\cos(\arctan u)$? *Hint*: use the triangle diagram introduced in class for trigonometric substitutions.



Solution. Consider the following triangle diagram:

We conclude that

$$\sin(\arctan u) = \sin\theta = \frac{u}{\sqrt{u^2+1}};$$
$$\cos(\arctan u) = \cos\theta = \frac{1}{\sqrt{u^2+1}}.$$

□

Problem 2.2 (4 points)**.** Set $u = \tan\frac{x}{2}$, so that $x = 2\arctan u$ and that $dx = \frac{2}{1+u^2}\,du$. What is $\sin x$? What is $\cos x$?

Solution. Using the double-angle formula

$$\sin 2\theta = 2\sin\theta\cos\theta,$$

we compute $\sin x$ as follows:

$$\sin x = \sin(2\arctan u) = 2\sin(\arctan u)\cos(\arctan u) = \frac{2u}{u^2+1}.$$

Similarly, using the double-angle formula

$$\cos 2\theta = \cos^2\theta - \sin^2\theta,$$

we compute $\cos x$ as follows:

$$\begin{aligned}\cos x = \cos(2\arctan u) &= \cos^2(\arctan u) - \sin^2(\arctan u)\\ &= \frac{1}{u^2+1} - \frac{u^2}{u^2+1} = \frac{1-u^2}{1+u^2}.\end{aligned}$$

□

Problem 2.3 (2 points)**.** Apply the tangent half-angle substitution on

$$\int \frac{1}{7\sin x - \cos x + 5}\,dx$$

and write the resulting integral. The answer should be of the form

$$\int \frac{1}{au^2+bu+c}\,du$$

for appropriate choices of a, b, and c.

Solution. Set $u = \tan\frac{x}{2}$, so that $x = 2\arctan u$ and that $dx = \frac{2}{1+u^2}\,du$. By the results of the previous problem, $\sin x = \frac{2u}{1+u^2}$ and $\cos x = \frac{1-u^2}{1+u^2}$.

$$\begin{aligned}\int \frac{1}{7\sin x - \cos x + 5}\,dx &= \int \frac{1}{\frac{14u}{1+u^2} + \frac{-1+u^2}{1+u^2} + \frac{5+5u^2}{1+u^2}} \cdot \frac{2}{1+u^2}\,du \\ &= \int \frac{1+u^2}{14u - 1 + u^2 + 5 + 5u^2} \cdot \frac{2}{1+u^2}\,du \\ &= \int \frac{2}{6u^2+14u+4}\,du = \int \frac{1}{3u^2+7u+2}\,du.\end{aligned}$$

□

Problem 2.4 (5 points)**.** Apply the method of partial fraction decomposition to compute the integral obtained in the previous problem. The answer should be of the form

$$a\ln|bu+c| + d\ln|eu+f| + C$$

for appropriate choices of a, b, c, d, e, f.

Solution. We use the method of partial fraction decomposition:

$$\frac{1}{3u^2+7u+2} = \frac{1}{(3u+1)(u+2)} = \frac{A}{3u+1} + \frac{B}{u+2}.$$

In order to compute A and B, we rewrite the above identity as follows:

$$\frac{1}{3u^2+7u+2} = \frac{A}{3u+1} + \frac{B}{u+2} = \frac{A(u+2)+B(3u+1)}{3u^2+7u+2}.$$

It then follows that

$$1 = A(u+2) + B(3u+1).$$

Substituting $u = -2$ into the above identity, we see that

$$1 = B(3\cdot -2 + 1) = -5B.$$

We conclude that $B = -\frac{1}{5}$. Similarly, setting $u = -\frac{1}{3}$ yields the identity

$$1 = A\left(-\frac{1}{3} + 2\right) = \frac{5}{3}A.$$

We conclude that $A = \frac{3}{5}$.

It follows that

$$\frac{1}{3u^2+7u+2} = \frac{5/3}{3u+1} + \frac{-1/5}{u+2},$$

whence

$$\begin{aligned}\int \frac{1}{3u^2+7u+2}\,du &= \frac{5}{3}\int \frac{1}{3u+1}\,du - \frac{1}{5}\int \frac{1}{u+2}\,du \\ &= \frac{5}{9}\ln|3u+1| - \frac{1}{5}\ln|u+2| + C.\end{aligned}$$

□

Problem 2.5 (2 points)**.** Substitute $u = \tan\frac{x}{2}$ back to the result obtained from the last integral to compute

$$\int \frac{1}{7\sin x - \cos x + 5}\,dx.$$

Solution. It suffices to observe that

$$\begin{aligned}\int \frac{1}{7\sin x - \cos x + 5}\,dx &= \int \frac{1}{3u^2+7u+2}\,du \\ &= \frac{5}{9}\ln|3u+1| - \frac{1}{5}\ln|u+2| + C \\ &= \frac{5}{9}\ln\left|3\tan\frac{x}{2}+1\right| - \frac{1}{5}\ln\left|\tan\frac{x}{2}+2\right| + C\end{aligned}$$

□

3. Bonus Problems (10 points)

Problem 3.1 (5 points)**.** Integrate $\int x^4 \arctan x\,dx$.

Solution. We integrate by parts: set $u = \arctan x$ and $dv = x^4\,dx$, so that $du = \frac{1}{1+x^2}\,dx$ and $v = \frac{1}{5}x^5$. Observe that

$$\int x^4 \arctan x\,dx = uv - \int v\,du = \frac{1}{5}x^5\arctan x - \frac{1}{5}\int \frac{x^5}{1+x^2}\,dx.$$

To compute $\int \frac{x^5}{1+x^2}$, we carry out polynomial long division:

$$\begin{array}{r|l} & x^3 - x \\ \hline x^2+1 & x^5 \\ & -x^5 - x^3 \\ \hline & -x^3 \\ & x^3 + x \\ \hline & x \end{array}$$

Therefore,

$$\int \frac{x^5}{1+x^2}\,dx = \int x^3 - x\,dx + \int \frac{x}{1+x^2}\,dx = \frac{1}{4}x^4 - \frac{1}{2}x^2 + \int \frac{x}{1+x^2}\,dx,$$

and so

$$\int x^4\arctan x\,dx = \frac{1}{5}x^5\arctan x - \frac{1}{20}x^4 + \frac{1}{10}x^2 - \frac{1}{5}\int\frac{x}{1+x^2}\,dx.$$

It therefore suffices to compute $\int \frac{x}{1+x^2}\,dx$. For this, we use the substitution $p = 1+x^2$, so that $dp = 2x\,dx$. We then see that

$$\int \frac{x}{1+x^2}\,dx = \frac{1}{2}\int\frac{1}{p}\,dp = \frac{1}{2}\ln|p| + C = \frac{1}{2}\ln|1+x^2| + C.$$

It now follows that

$$\int x^4\arctan x\,dx = \frac{1}{5}x^5\arctan x - \frac{1}{20}x^4 + \frac{1}{10}x^2 - \frac{1}{10}\ln|1+x^2| + C.$$

□

Problem 3.2 (5 points)**.** Find the area enclosed by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Solution. See p.323 of the course textbook. □