CALCULUS II, SUMMER 2015 - WEEKEND PROBLEM SET 1

`60 points total = 50 points + 10 extra credit points`

`Name:` ____ `Score:` ______/ 50

Use this worksheet as the cover sheet for your write-up: write your name on this page, and staple this sheet to the front of your homework packet.

Please indicate clearly which problems you have worked on. You will receive no credit for submitting solutions that the grader cannot read and understand—be sure to write legibly!

1. The Logarithm (34 points)

Problem 1.1 (10 points)**.** For each $x > 0$, the natural logarithm of x is defined to be the integral

$$\ln x = \int_1^x \frac{1}{t}\,dt.$$

(1) Explain why $\ln 1 = 0$.
(2) Explain why $\frac{d}{dx}\ln x = \frac{1}{x}$ for all $x > 0$.
(3) Explain why $\frac{d}{dx}\ln|x| = \frac{1}{x}$ for all $x \neq 0$. *Hint*: Apply the chain rule, and consider the $x > 0$ case and the $x < 0$ case separately. Conclude that $\int \frac{1}{x}\,dx = \ln|x| + C$.
(4) Explain why $\ln ab = \ln a + \ln b$ for all $a > 0$ and $b > 0$. *Hint*: $\int_1^{ab} \frac{1}{t}\,dt = \int_1^{a} \frac{1}{t}\,dt + \int_a^{ab} \frac{1}{t}\,dt$.
(5) Explain why $\ln x^r = r\ln x$ for all $r > 0$ and $x > 0$.
Hint: Define $f(x) = \ln x^r - r\ln x$. differentiate f using (2), and apply (1) to show that $f(x) = 0$ for all x. Conclude that $\ln x^r = r\ln x$.

Problem 1.2 (2 points)**.** Let f be a differentiable function with a continuous derivative f'. What is $\int \frac{f'(x)}{f(x)}\,dx$?

Problem 1.3 (4 points)**.** Integrate $\int \tan x\,dx$ and $\int \cot x\,dx$.

Problem 1.4 (6 points)**.** Integrate $\int \sec x\,dx$ and $\int \csc x\,dx$.

Problem 1.5 (6 points)**.** Integrate $\int \sin(\ln x)\,dx$ and $\int \cos(\ln x)\,dx$.

Problem 1.6 (6 points)**.** Given two integers $m \geq 1$ and $n \geq 1$, derive the reduction formula

$$\int x^m \ln^n x\,dx = \frac{x^{m+1}\ln^n x}{m+1} - \frac{n}{m+1}\int x^m \ln^{n-1} x\,dx.$$

Use the formula to integrate $\int x^{10}\ln^3 x\,dx$.

Date: July 9, 2015.

2. Tangent half-angle substitution (16 points)

The substitution technique we study in this section is what Michael Spivak, a legendary expositor of mathematics, calls the "world's sneakiest substitution."

If we set $u = \tan \frac{x}{2}$, then $x = 2 \arctan u$, so that $dx = \frac{2}{1+u^2}\, du$. The substitution is used to transform the integral of a function obtained by combining sines and cosines via addition, multiplication, and division (a "rational function of sines and cosines") into the integral of a rational function. The latter integral can then be computed by the method of partial fraction decomposition, integration by parts, and integration by substitution.

Problem 2.1 (3 points)**.** What is $\sin(\arctan u)$? What is $\cos(\arctan u)$? *Hint*: use the triangle diagram introduced in class for trigonometric substitutions.

Problem 2.2 (4 points)**.** Set $u = \tan \frac{x}{2}$, so that $x = 2 \arctan u$ and that $dx = \frac{2}{1+u^2}\, du$. What is $\sin x$? What is $\cos x$?

Problem 2.3 (2 points)**.** Apply the tangent half-angle substitution on

$$\int \frac{1}{7 \sin x - \cos x + 5}\, dx$$

and write the resulting integral. The answer should be of the form

$$\int \frac{1}{au^2 + bu + c}\, du$$

for appropriate choices of a, b, and c.

Problem 2.4 (5 points)**.** Apply the method of partial fraction decomposition to compute the integral obtained in the previous problem. The answer should be of the form

$$a \ln |bu + c| + d \ln |eu + f| + C$$

for appropriate choices of a, b, c, d, e, f.

Problem 2.5 (2 points)**.** Substitute $u = \tan \frac{x}{2}$ back to the result obtained from the last integral to compute

$$\int \frac{1}{7 \sin x - \cos x + 5}\, dx.$$

3. Bonus Problems (10 points)

Problem 3.1 (5 points)**.** Integrate $\int x^4 \arctan x\, dx$.

Problem 3.2 (5 points)**.** Find the area enclosed by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$