CALCULUS II, SUMMER 2015 - WEEKEND PROBLEM SET 2

1. The "Paradox" of Gabriel's Horn (10 points)

Let $f(x) = \frac{1}{x}$.

Problem 1.1 (2 points)**.** Sketch the solid obtained by rotating f with respect to the x-axis on $[1, t]$. The infinitely-long horn-shaped object obtained by sending $t \to \infty$ is called *Gabriel's horn*.

Problem 1.2 (4 points)**.** What is the volume of Gabriel's horn?

Solution.

$$\int_1^\infty \pi[f(x)]^2\,dx = \int_1^\infty \frac{\pi}{x^2}\,dx = \lim_{x\to\infty} -\frac{\pi}{x} + \pi = \pi.$$

□

Problem 1.3 (4 points)**.** What is the surface area of Gabriel's horn?

Solution. Since

$$\sqrt{1 + \frac{1}{x}^4} \geq 1$$

for all $x \geq 1$, we see that

$$\begin{aligned}\int_1^\infty 2\pi f(x)\sqrt{1 + [f'(x)]^2}\,dx &= 2\pi \int_1^\infty \frac{\sqrt{1 + \frac{1}{x^4}}}{x}\,dx \\ &\geq 2\pi \int_1^\infty \frac{1}{x}\,dx = \lim_{x\to\infty} 2\pi \ln x = \infty.\end{aligned}$$

By the comparison theorem,

$$\int_1^\infty 2\pi f(x)\sqrt{1 + [f'(x)]^2}\,dx = \infty.$$

□

2. Exam 1: A Look Back (50 points)

See the solutions for Exam 1.

Date: July 20, 2015.