CALCULUS II, SUMMER 2015 - WEEKEND PROBLEM SET 3 SOLUTIONS

`60 points total = 50 points + 10 extra credit points`

1. Radioactive Decay (10 points)

Let $y = f(t)$ denote the amount of radioactive particles present at time t. The *universal law of radioactive decay* states that

$$y' = -ky, \tag{1.1}$$

where the *decay constant* k is determined by the type of particles in question.

Problem 1.1 (2 points)**.** What is the solution of (1.1), given the initial condition $y(0) = y_0$?

Solution. The general solution of (1.1) is

$$y = Ce^{-kx},$$

where C is a constant to be determined. Since $y(0) = y_0$, we see that

$$y_0 = y(0) = Ce^{-k\cdot 0} = C,$$

It follows that

$$y = y_0 e^{-kx}.$$

□

The *half-life* of a radioactive element is the unique number t_H such that

$$\frac{y(t_H)}{y(0)} = \frac{1}{2}.$$

Problem 1.2 (2 points)**.** What is t_H in terms of k?

Solution. Observe that

$$\frac{1}{2} = \frac{y(t_H)}{y(0)} = \frac{y_0 e^{-kt_H}}{y_0 e^{-k\cdot 0}} = e^{-kt_H}.$$

Therefore,

$$\ln\frac{1}{2} = -kt_H,$$

and so

$$t_H = \frac{\ln 2}{k}.$$

□

Problem 1.3 (2 points)**.** Show that

$$\frac{y(t+t_H)}{y(t)} = \frac{y(t_H)}{y(0)}$$

for all t.

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Solution. Observe that
$$\frac{y(t+t_H)}{y(t)} = \frac{y_0 e^{-k(t+t_H)}}{y_0 e^{-kt}} = \frac{y_0 e^{-kt_H} e^{-kt}}{y_0 e^{-kt}} = \frac{y_0 e^{-kt_H}}{y_0} = \frac{y(t_H)}{y(0)}.$$
□

Problem 1.4 (2 points)**.** Fix a decay constant k. Let t_H be the half-life obtained from (1.1) with initial condition $y = y_0$. Similarly, let $\widetilde{t_H}$ be the half-life obtained from (1.1) with initial condition $y = y_1$. Show that $t_H = \widetilde{t_H}$.

Solution. By Problem 1.2,
$$t_H = \frac{\ln 2}{k} = \widetilde{t_H}.$$
□

Problem 1.5 (2 points)**.** The half-life for Polonium-210 is approximately 140 days. Find what percentage of a given quantity of Polonium-210 remains after 1 year. (*Hint*: Use a calculator.)

Solution. By Problem 1.2,
$$140 = \frac{\ln 2}{k},$$
and so
$$k = \frac{\ln 2}{140}.$$
Given 100 units of Polonium-210, we see that
$$y(365) = 100e^{-\frac{\ln 2}{140}\cdot 365} \approx 16.4$$
units of Polonium-210 remain after 1 year. It follows that approximately 16.4% of a given quantity of Polonium-210 remains after 1 year. □

2. Free Fall (15 points)

Recall that the *displacement function* $s(t)$ of an object gives the distance, at time t, between the object and the starting point. The *velocity function* $v(t)$ is given by the formula
$$v(t) = s'(t),$$
and the *acceleration function* $a(t)$ is given by the formula
$$a(t) = s''(t).$$

Suppose that we drop an object of mass m in the Earth's atmosphere. The two major forces acting on the body are the gravitational force mg, where g is the gravitational constant determined by the location on Earth, and air resistance kv, where the air resistance constant k is determined by various physical factors that we do not wish to discuss here. For the sake of simplicity, we ignore all other forces in our model.

The gravitational force pulls the object to the ground, and the air resistance pushes the object away from the earth, and so the net force exerted on the object is
$$F = mg - kv.$$
By Newton's second law $F = ma$,
$$ma = mg - kv.$$

Since $a = v'$, we obtain the differential equation

$$mv' = mg - kv. \tag{2.1}$$

Problem 2.1 (5 points)**.** Solve (2.1) with the initial value $v(0) = 0$, and compute the displacement function $s(t)$.

Solution. Note that (2.1) is equivalent to

$$v' + \frac{k}{m}v = g,$$

which is a nonhomogeneous first-order linear ODE. Therefore,

$$v = be^{-A(t)} + e^{-A(t)} \int_a^t e^{A(u)} Q(u)\, du,$$

where $a = 0$, $b = 0$, $P(t) = k/m$, $Q(t) = g$, and $A(t) = \int_a^t P(x)\, dx$. Quick computations yield

$$v(t) = \frac{mg}{k}(1 - e^{-kt/m}).$$

Computing the displacement function $s(t)$ amounts to integrating $v(t)$ with respect to t:

$$s(t) = \int v(t)\, dt + C = \frac{mg}{k}t + \frac{m^2 g}{k^2}e^{-kt/m} + C.$$

By definition, $s(0) = 0$, and so

$$C = s(0) - \frac{mg}{k} \cdot 0 - \frac{m^2 g}{k^2}e^{-(k \cdot 0)/m} = -\frac{m^2 g}{k^2}.$$

It follows that

$$s(t) = \frac{mg}{k}t + \frac{m^2 g}{k^2}(e^{-kt/m} - 1).$$

□

Problem 2.2 (5 points)**.** Solve (2.1) with the initial value $v(0) = v_0$, and compute the displacement function $s(t)$.

Solution. Note that (2.1) is equivalent to

$$v' + \frac{k}{m}v = g,$$

which is a nonhomogeneous first-order linear ODE. Therefore,

$$v = be^{-A(t)} + e^{-A(t)} \int_a^t e^{A(u)} Q(u)\, du,$$

where $a = 0$, $b = v_0$, $P(t) = k/m$, $Q(t) = g$, and $A(t) = \int_a^t P(x)\, dx$. Quick computations yield

$$v(t) = \frac{mg}{k}(1 - e^{-kt/m}) + v_0 e^{-kt/m}.$$

Computing the displacement function $s(t)$ amounts to integrating $v(t)$ with respect to t:

$$s(t) = \int v(t)\, dt + C = \frac{mg}{k}t + \frac{m^2 g}{k^2}e^{-kt/m} - \frac{mv_0}{k}e^{-kt/m} + C.$$

By definition, $s(0) = 0$, and so

$$C = s(0) - \frac{mg}{k} \cdot 0 - \frac{m^2 g}{k^2} e^{-(k \cdot 0)/m} + \frac{mv_0}{k} e^{-(k \cdot 0)/m} = -\frac{m^2 g}{k^2} + \frac{mv_0}{k}.$$

It follows that

$$s(t) = \frac{mg}{k} t + \frac{m^2 g}{k^2}(e^{-kt/m} - 1) + \frac{mv_0}{k}(1 - e^{-kt/m}).$$

□

Problem 2.3 (2 points)**.** The *terminal velocity* of an object in free fall is defined to be the limit

$$v_T = \lim_{t \to \infty} v(t).$$

Compute the terminal velocity of an object in free fall with the initial velocity $v(0) = v_0$. Does the terminal velocity change depending on the value of v_0?

Solution. By Problem 2.2,

$$v_T = \lim_{t \to \infty} v(t) = \lim_{t \to \infty} \frac{mg}{k}(1 - e^{-kt/m}) + v_0 e^{-kt/m} = \frac{mg}{k}.$$

It follows that v_T is invariant of the choice of v_0. □

Problem 2.4 (3 points)**.** An object of mass 1000 kg is thrown downwards from a helicopter at altitude 5000m. If the initial velocity of the object is 10 m/s, what is the altitude of the object after 1 minute? How about the velocity? For simplicity's sake, use $g = 10$ and $k = 100$. (*Hint*: use a calculator.)

Solution. By Problem 2.2,

$$s(t) = 100t + 1000(e^{-t/10} - 1) + 100(1 - e^{-t/10})$$

Since

$$s(60) \approx 5102,$$

the object will have already hit the ground. Therefore, the altitude of the object is 0m, and the velocity of the object is 0m/s. □

3. Simple Harmonic Motion

Recall that the *displacement function* $s(t)$ of an object gives the distance, at time t, between the object and the starting point. The *velocity function* $v(t)$ is given by the formula

$$v(t) = s'(t),$$

and the *acceleration function* $a(t)$ is given by the formula

$$a(t) = s''(t).$$

Consider a spring hooked to the ceiling with an object of mass m attached to the bottom of it. *Hooke's law*[1]

$$F = -ks,$$

[1]Hooke's law is obtained by considering the first-order term of the Taylor approximation of the restoring force. By the physical assumptions, the zeroth term is zero, and the higher-order terms as suitably small.

where the *spring constant* $k > 0$ depends on the type of spring, describes the restoring force of the spring; here we consider the displacement of the bottom end of the spring. By Newton's second law $F = ma$,

$$ma = -ks.$$

Since $a = s''$, we obtain the *motion equation for simple harmonic motion*

$$ms'' = -ks. \tag{3.1}$$

Problem 3.1 (5 points)**.** Set $\omega = \sqrt{\frac{k}{m}}$, so that (3.1) can be written as follows:

$$s'' = -\omega^2 s.$$

Show that the general solution of (3.1) is of the form

$$s = c_1 \cos(\omega t) + c_2 \sin(\omega t)$$

for arbitrary constants c_1 and c_2. Show that the above solution can be written as

$$s = A\cos(\omega t - \varphi),$$

where

$$A = \sqrt{c_1^2 + c_2^2} \qquad \text{and} \qquad \varphi = \arctan\frac{c_2}{c_1}.$$

We say that A is the *amplitude* of the motion, ω the *angular frequency* of the motion, and φ the *phase* of the motion.

Solution. It is not hard to check that $\cos(\omega t)$ and $\sin(\omega t)$ are linearly independent solutions of (3.1). Since (3.1) is a homogeneous second-order ODE with constant coefficients, every solution of (3.1) can be written as a linear combination of these two solutions. Therefore, the general solution is of the form

$$s = c_1 \cos(\omega t) + c_2 \sin(\omega t).$$

We now fix c_1 and c_2 and set

$$A = \sqrt{c_1^2 + c_2^2} \qquad \text{and} \qquad \varphi = \arctan\frac{c_2}{c_1}.$$

Since

$$\sin(\arctan\theta) = \frac{\theta}{\sqrt{1+\theta^2}} \qquad \text{and} \qquad \cos(\arctan\theta) = \frac{1}{\sqrt{1+\theta^2}},$$

we see that

$$\begin{aligned} c_1\cos(\omega t) + c_2\sin(\omega t) &= A\left(\frac{c_1}{\sqrt{c_1^2+c_2^2}}\cos(\omega t) + \frac{c_2}{\sqrt{c_1^2+c_2^2}}\sin(\omega t)\right) \\ &= A\left(\cos\varphi\cos(\omega t) + \sin\varphi\sin(\omega t)\right) \\ &= A\cos(\omega t - \varphi), \end{aligned}$$

as was to be shown. □

Problem 3.2 (10 points)**.** Consider a simple harmonic oscillator. Suppose that its initial displacement is 1, its initial velocity is 2, and its initial acceleration is -12. Compute the displacement and acceleration of the simple harmonic oscillator when the velocity is $\sqrt{8}$.

Solution. By Problem 3.1,
$$s = A\cos(\omega t - \varphi).$$
Since $s(0) = 1$, we see that
$$1 = s(0) = A\cos(-\varphi),$$
and so
$$-\varphi = \arccos\frac{1}{A}.$$
Now,
$$v = s'(t) = -\omega A\sin(\omega t - \varphi),$$
Since $v(0) = 2$, we see that
$$2 = v(0) = -\omega A\sin\left(\arccos\frac{1}{A}\right) = -\omega A\sqrt{1 - \frac{1}{A^2}} = -\omega\sqrt{A^2 - 1}.$$
Moreover,
$$a = v'(t) = -\omega^2 A\cos(\omega t - \varphi),$$
and so
$$-12 = a(0) = -\omega^2 A\cos(-\varphi) = -\omega^2.$$
Therefore,
$$\omega = -\sqrt{12},$$
and so
$$\sqrt{A^2 - 1} = \frac{2}{-\omega} = \frac{1}{\sqrt{3}}.$$
It follows that
$$A^2 = \frac{4}{3},$$
and so
$$A = \frac{2}{\sqrt{3}}$$
and
$$\varphi = -\arccos\frac{\sqrt{3}}{2} = -\frac{\pi}{6}.$$

We conclude that
$$s(t) = \frac{2}{\sqrt{3}}\cos\left(-\sqrt{12}t + \frac{\pi}{6}\right);$$
$$v(t) = -4\sin\left(-\sqrt{12}t + \frac{\pi}{6}\right);$$
$$a(t) = \frac{24}{\sqrt{3}}\cos\left(-\sqrt{12}t + \frac{\pi}{6}\right).$$

Now, $v(t) = \sqrt{8}$ when
$$\sin\left(-\sqrt{12}t + \frac{\pi}{6}\right) = -\frac{\sqrt{2}}{2}.$$
This happens when, for example,
$$-\sqrt{12}t + \frac{\pi}{6} = \frac{5\pi}{4}.$$
In this case,
$$s(t) = \frac{2}{\sqrt{3}}\cos\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{\sqrt{3}} \quad \text{and} \quad a(t) = \frac{24}{\sqrt{3}}\cos\left(\frac{5\pi}{4}\right) = -4\sqrt{6}.$$

□

4. Damped Harmonic Oscillator (20 points)

Oscillators are often subjected to various external forces. As in Section 2, we shall consider just one additional force: air resistance[2] $cv = cs'$. The adding of external force results in the following modified form of (3.1):

$$ms'' + cs' + ks = 0.$$

Letting $\omega = \sqrt{\frac{k}{m}}$ and $\zeta = \frac{c}{2m\omega}$, we can rewrite the above equation to obtain the *motion equation for damped harmonic oscillation*:

$$s'' + 2\zeta\omega s' + \omega^2 s = 0. \tag{4.1}$$

Problem 4.1 (5 points)**.** ζ in (4.1) is called the *damping constant.*

(1) Solve (4.1) when $\zeta > 1$. This case is referred to as *overdamped* and models the situation where the system reaches an equilibrium without oscillating.
(2) Solve (4.1) when $\zeta = 1$. This case is referred to as *critically damped* and models the situation where the system reaches an equilibrium as rapidly as possible without oscillating.
(3) Solve (4.1) when $\zeta < 1$. This case is referred to as *underdamped* and models the situation where the system oscillates with gradually decreasing amplitude.

Solution. (4.1) is a homogeneous second-order ODE with constant coefficients, so the general solution depends on the value of the discriminant

$$d = 4\omega^2 - (2\zeta\omega)^2 = 4\omega^2(1 - \zeta^2).$$

If $\zeta > 1$, then $d < 0$, and so

$$s = e^{-\zeta\omega t}\left(c_1 e^{\omega\sqrt{\zeta^2-1}t} + c_2 e^{-\omega\sqrt{\zeta^2-1}t}\right).$$

If $\zeta = 1$, then $d = 0$, and so

$$s = e^{-\zeta\omega t}\left(c_1 + c_2 t\right).$$

If $\zeta < 1$, then $d > 0$ and so

$$s = e^{-\zeta\omega t}\left(c_1 \sin\left(\omega\sqrt{1-\zeta^2}t\right) + c_2\cos\left(\omega\sqrt{1-\zeta^2}t\right)\right).$$

□

If the system is underdamped, then the general form of the displacement function is of the form

$$s = e^{at}\left(c_1\sin(bt) + c_2\cos(bt)\right)$$

for appropriately chosen constants a and b. The *maximum amplitude* of the oscillator is

$$A_{\max}(t) = e^{at}\sqrt{c_1^2 + c_2^2},$$

which should decrease over time. The *damped angular frequency* of the oscillator is

$$\omega_1 = b.$$

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[2] While we have chosen to incorporate air resistance, the damped harmonic oscillator model can account for other types of external forces, such as friction.

Problem 4.2 (15 points)**.** A damped harmonic oscillator consists of an object of mass 100 attached to a spring whose spring constant is 10000. The initial displacement of the oscillator is 3 and the initial velocity is 0. Suppose that the maximum amplitude of the oscillator decreased to half the initial value after 10 seconds of oscillation. Compute the damping constant and the damped angular frequency of the oscillator.

Solution. Since the amplitude decreases over time, the damped harmonic oscillator in question is underdamped. Therefore, the general form of the displacement function is given by

$$s(t) = e^{-\zeta\omega t}\left(c_1 \sin\left(\omega\sqrt{1-\zeta^2}t\right) + c_2 \cos\left(\omega\sqrt{1-\zeta^2}t\right)\right).$$

Since $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{10000}{100}} = 10$, we see that

$$s(t) = e^{-10\zeta t}\left(c_1 \sin\left(10\sqrt{1-\zeta^2}t\right) + c_2 \cos\left(10\sqrt{1-\zeta^2}t\right)\right).$$

Note that the maximum amplitude of the oscillator is given by

$$A_{\max}(t) = e^{-10\zeta t}\sqrt{c_1^2 + c_2^2}.$$

Since

$$\frac{1}{2} = \frac{A_{\max}(10)}{A_{\max}(0)} = \frac{e^{-100\zeta}\sqrt{c_1^2+c_2^2}}{1\sqrt{c_1^2+c_2^2}} = e^{-100\zeta},$$

we see that

$$\zeta = \frac{\ln 2}{100}.$$

It then follows that the damped angular frequency of the oscillator is

$$10\sqrt{1-\zeta^2} = 10\sqrt{1-\left(\frac{\ln 2}{100}\right)^2}.$$

□