CALCULUS II, SUMMER 2015 - WEEKEND PROBLEM SET 4

`60 points total = 50 points + 10 extra credit points`

Read Section 9.5, the section on conic sections, in Stewart, and do the following problems.

Problem 1 (20 points)**.** Write a polar equation of a conic with the focus at the origin and the given data:

1. Parabola, directrix $x = -3$
2. Hyperbola, eccentricity 3, directrix $x = 3$
3. Ellipse, eccentricity 0.8, vertex $(1, \pi/2)$
4. Hyperbola, eccentricity 3, directrix $r = -6 \csc\theta$

Solutions. 1. $r = \frac{3}{1-\cos\theta}$

2. $r = \frac{9}{1+3\cos\theta}$

3. Since $(r, \theta) = (1, \pi/2)$ is a vertex, the vertices of the ellipse in question are on the y-axis. Therefore, the desired equation of the ellipse is of the form

$$r = \frac{ed}{1 \pm e\sin\theta}.$$

$e = 0.8$, and so the equation can be written as follows:

$$r = \frac{4d}{5 \pm 4\sin\theta}.$$

Since we have already used all of the information given to us, we conclude that there are two possible equations:

$$r = \frac{4d}{5 + 4\sin\theta} \quad \text{and} \quad r = \frac{4d}{5 - 4\sin\theta}.$$

We determine d in each case. If we choose $+$, then

$$1 = r = \frac{4d}{5 + 4\sin(\pi/2)} = \frac{4d}{9},$$

and so $d = 9/4$. If we choose $-$, then

$$1 = r = \frac{4d}{5 - 4\sin(\pi/2)} = \frac{4d}{1},$$

and so $d = 1/4$. Therefore, the two possible equations are

$$r = \frac{9}{5 + 4\sin\theta} \quad \text{and} \quad r = \frac{1}{5 - 4\sin\theta}.$$

4. Since

$$\csc\theta = \csc(\arctan(y/x)) = \frac{1}{y}\sqrt{x^2 + y^2},$$

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we see that $r = -\csc\theta$ is equivalent to

$$y = -6.$$

Therefore, the desired equation is of the form

$$r = \frac{ed}{1 - e\sin\theta},$$

where $d = 6$. Since $e = 3$, we conclude that

$$r = \frac{18}{1 - 3\sin\theta}.$$

□

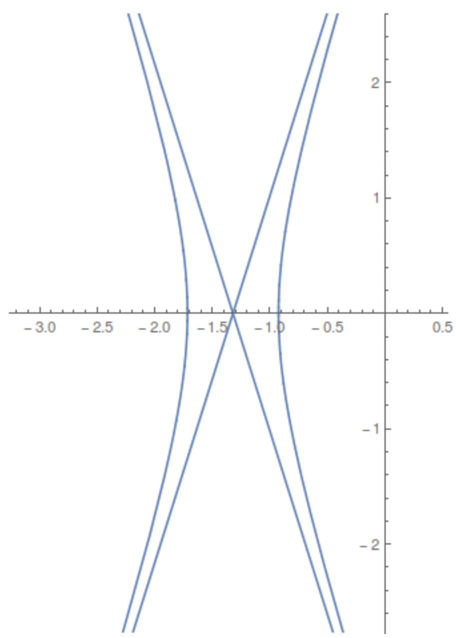
Problem 2 (20 points)**.** For each of the following polar curves, (a) find the eccentricity, (b) identify the conic, (c) give an equation of the directrix, and (d) sketch the conic.

1. $r = \frac{12}{3-10\cos\theta}$
2. $r = \frac{3}{2+2\cos\theta}$
3. $r = \frac{5}{2-2\sin\theta}$
4. $r = \frac{4}{2+\cos\theta}$

Solutions. 1. Observe that

$$r = \frac{12}{3 - 10\cos\theta} = \frac{4}{1 - (10/3)\cos\theta}.$$

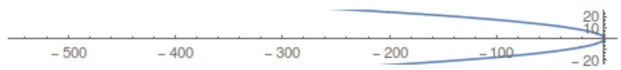
Therefore, $e = 10/3$, $d = 6/5$, the equation of the directrix is $x = -6/5$, and the equation describes a hyperbola.



2. Observe that

$$r = \frac{3}{2 + 2\cos\theta} = \frac{3/2}{1 + 1\cos\theta}.$$

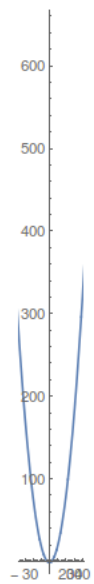
Therefore, $e = 1$, $d = 3/2$, the equation of the directrix is $x = 3/2$, and the equation describes a parabola.



3. Observe that

$$r = \frac{5}{2 - 2\sin\theta} = \frac{5/2}{1 - 1\sin\theta}.$$

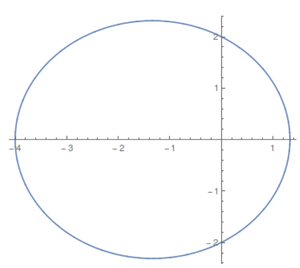
Therefore, $e = 1$, $d = 5/2$, the equation of the directrix is $y = -5/2$, and the equation describes a hyperbola.



4. Observe that

$$r = \frac{4}{2 + \cos\theta} = \frac{2}{1 + (1/2)\cos\theta}.$$

Therefore, $e = 1/2$, $d = 4$, the equation of the directrix is $x = 4$, and the equation describes an ellipse.



□

Problem 3 (10 points)**.** The Hale–Bopp coment, discovered in 1995, has an elliptical orbit with eccentricity 0.9951 and the length of the major axis is 356.5 AU. Find a polar equation for the orbit of this comet. How close to the sun does it come?

Solution. Recall that the equation of the ellipse with a focus at the origin, eccentricity e, and directrix $x = -d$ is

$$r = \frac{ed}{1 - e\cos\theta}.$$

If the length of the major axis of the ellipse is $2a$, then the length of the semi-major axis is a. Since the length of the semi-major axis is the average of the shortest distance

$$r_{\text{min}} = \frac{ed}{1+e}$$

between the focus (the origin) and the ellipse and the longest distance

$$r_{\text{max}} = \frac{ed}{1-e}$$

between the focus (the origin) and the ellipse, we see that

$$a = \frac{r_{\text{min}} + r_{\text{max}}}{2} = \frac{\frac{ed(1-e}{+} ed(1+e)}{2(1-e^2)} = \frac{ed}{1-e^2}.$$

Therefore,

$$d = \frac{a(1-e^2)}{e},$$

and the equation of the ellipse can be written as follows:

$$r = \frac{a(1-e^2)}{1 - e\cos\theta}.$$

For the Hale–Bopp comet problem, we have $e = 0.9951$ and $2a = 365.6$, and so the equation of the ellipse is

$$r = \frac{1.787}{1 - 0.9951\cos\theta}.$$

We obtain the minimum value of r when $\cos\theta = -1$, and so

$$r_{\text{min}} \approx \frac{1.787}{1 + 0.9951} \approx 0.8957\ AU.$$

□

Problem 4 (10 points)**.** Show that the parabolas $r = c/(1+\cos\theta)$ and $r = d/(1-\cos\theta)$ intersect at right angles.

Proof. Two curves intersect at right angles at a point P if their tangent lines at point P form a right angle. This happens if and only if the product of the slopes of the tangent lines is -1. We let

$$r_1 = \frac{c}{1+\cos\theta} \quad \text{and} \quad r_2 = \frac{d}{1-\cos\theta}$$

and compute $\frac{dy}{dx}$ of both curves. For this, we recall that

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}.$$

For the first curve,

$$\begin{aligned}\frac{dy_1}{dx} &= \frac{\frac{c\sin^2\theta}{(1+\cos\theta)^2} + \frac{c\cos\theta}{1+\cos\theta}}{\frac{c\sin\theta\cos\theta}{(1+\cos\theta)^2} - \frac{c\sin\theta}{1+\cos\theta}} \\ &= \frac{\sin^2\theta + \cos\theta(1+\cos\theta)}{\sin\theta\cos\theta - \sin\theta(1+\cos\theta)} \\ &= \frac{\cos\theta + 1}{-\sin\theta}\end{aligned}$$

For the second curve,

$$\begin{aligned}\frac{dy_2}{dx} &= \frac{-\frac{d\sin^2\theta}{(1-\cos\theta)^2} + \frac{d\cos\theta}{1-\cos\theta}}{-\frac{d\sin\theta\cos\theta}{(1-\cos\theta)^2} - \frac{d\sin\theta}{1-\cos\theta}} \\ &= \frac{-\sin^2\theta + \cos\theta(1-\cos\theta)}{-\sin\theta\cos\theta - \sin\theta(1-\cos\theta)} \\ &= \frac{\cos\theta - 1}{-\sin\theta}.\end{aligned}$$

Therefore,

$$\frac{dy_1}{dx}\cdot\frac{dy_2}{dx} = \frac{(\cos\theta+1)(\cos\theta-1)}{\sin^2\theta} = \frac{\cos^2\theta - 1}{\sin^2\theta}.$$

It now follows that $\left.\frac{dy_1}{dx}\cdot\frac{dy_2}{dx}\right|_{\theta=0} = -1.$ □

All problems are taken from page 534 of James Stewart, *Essential Calculus: Early Transcendentals* (2e).