CALCULUS II, SUMMER 2015 - WEEKEND PROBLEM SET 5 SOLUTIONS

`60 points total = 50 points + 10 extra credit points`

Problem 1 (10 points)**.** For each integer $n \geq 1$, we let

$$a_{2n} = a_{2n+1} = \frac{1}{2^n}.$$

Apply the ratio test and the root test on the series

$$\sum_{n=0}^{\infty} a_n.$$

What are the results?

Solution. Observe that

$$\frac{a_{n+1}}{a_n} = \begin{cases} 1 & \text{if } n \text{ is even;} \\ \frac{1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

Therefore,

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

does not exist.

On the other hand, we observe that

$$\sqrt[2n]{|a_{2n}|} = \left(\frac{1}{2^n}\right)^{1/2n} = 2^{-\frac{1}{2}}$$

$$\sqrt[2n+1]{|a_{2n+1}|} = \left(\frac{1}{2^n}\right)^{1/(2n+1)} = 2^{-\frac{n}{2n+1}}$$

From this, it is reasonable to claim that that[1]

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = 2^{-\frac{1}{2}}.$$

Proof using the formal definition of limit. To verify the claim, we first observe that

$$\lim_{n\to\infty} 2^{-\frac{n}{n+1}} = 2^{\left(\lim_{n\to\infty} -\frac{n}{n+1}\right)} = 2^{-\frac{1}{2}}. \tag{1.1}$$

Fix $\varepsilon > 0$, and use the formal definition of sequential limit to find a positive integer N_0 such that $k \geq N_0$ implies

$$\left| 2^{-\frac{k}{k+1}} - 2^{-\frac{1}{2}} \right| < \varepsilon. \tag{1.2}$$

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[1] Give full credit if the write-up includes the correct limit but provides no formal proof involving the rigorous definition of sequential limit.

Now, we set $N = 2N_0 + 1$. Suppose that $n \geq N$, and write $n = 2k + i$, where k is a positive integer and i is either 0 or 1. If $i = 0$, then (1.1) implies that

$$|\sqrt[n]{|a_n|} - 2^{-\frac{1}{2}}| = |\sqrt[n]{a_{2k}} - 2^{-\frac{1}{2}}| = |2^{-\frac{1}{2}} - 2^{-\frac{1}{2}}| = 0 < \varepsilon.$$

If $i = 1$, then $2k + i = n \geq N = 2N_0 + 1$, and so $k \geq N_0$. Therefore, (1.1) and (1.2) imply that

$$|\sqrt[n]{|a_n|} - 2^{-\frac{1}{2}}| = |\sqrt[n]{a_{2k+1}} - 2^{-\frac{1}{2}}| = |2^{-\frac{k}{2k+1}} - 2^{-\frac{1}{2}}| < \varepsilon.$$

It follows that

$$|\sqrt[n]{|a_n|} - 2^{-\frac{1}{2}}| < \varepsilon$$

for all $n \geq N$. Since the choice of $\varepsilon > 0$ was arbitrary, we conclude that

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = 2^{-\frac{1}{2}}.$$

converges. □

Proof using the squeeze theorem. Observe that

$$\sqrt[2n+1]{|a_{2n+1}|} = 2^{-\frac{n}{2n+1}} = 2^{-\frac{n+1/2}{2n+1} + \frac{1/2}{2n+1}} = 2^{-\frac{1}{2} + \frac{1}{4n+2}} = 2^{-\frac{1}{2}} \cdot 2^{\frac{1}{2(2n+1)}}.$$

Observe also that

$$\sqrt[2n]{|a_{2n}|} = 2^{-\frac{1}{2}} \leq 2^{-\frac{1}{2}} \cdot 2^{\frac{1}{2(2n)}},$$

as $2^{\frac{1}{2(2n)}} \geq 1$ for all $n \geq 1$. It follows that

$$\sqrt[n]{|a_n|} \leq 2^{-\frac{1}{2}} \cdot 2^{\frac{1}{2n}}$$

regardless of our choice of n.

Conversely, the inequality $2^{\frac{1}{2(2n)}} \geq 1$ also implies that

$$\sqrt[2n+1]{|a_{2n+1}|} = 2^{-\frac{1}{2}} \cdot 2^{\frac{1}{2(2n+1)}} \geq 2^{-\frac{1}{2}}.$$

Since

$$\sqrt[2n]{|a_{2n}|} = 2^{-\frac{1}{2}}$$

for all $n \geq 1$, it follows that

$$\sqrt[n]{|a_n|} \geq 2^{-\frac{1}{2}}$$

for all $n \geq 1$.

Now,

$$2^{-\frac{1}{2}} \leq \sqrt[n]{a_n} \leq 2^{-\frac{1}{2}}.$$

for all $n \geq 1$, and

$$\lim_{n\to\infty} 2^{-\frac{1}{2}} = \lim_{n\to\infty} 2^{-\frac{1}{2}} \cdot 2^{\frac{1}{2n}} = 2^{-\frac{1}{2}},$$

and so the squeeze theorem implies that

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = 2^{-\frac{1}{2}}.$$

□

Either way, we have that

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = 2^{-\frac{1}{2}}.$$

Since

$$2^{-\frac{1}{2}} < 1,$$

the root test implies that

$$\sum_{n=0}^{\infty} a_n$$

$\square$

Problem 2 (20 points)**.** Solve the following problems:

(1) For each integer $n \geq 1$, show that

$$0 \leq \frac{1}{n} - \int_n^{n+1} \frac{1}{x}\,dx \leq \frac{1}{n} - \frac{1}{n+1}.$$

(2) Use (1) to show that

$$\sum_{n=1}^{N} \frac{1}{n} - \int_1^N \frac{1}{x}\,dx \leq \frac{1}{N} + \sum_{n=1}^{N-1} \frac{1}{n^2+n}$$

for each $N \geq 2$.

(3) Show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2+n}$$

converges.

(4) Use (1), (2), and (3) to show that

$$\lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{n} - \int_1^N \frac{1}{x}\,dx$$

converges. The limit is called the *Euler–Mascheroni constant* and is often denoted by γ.

Solution to Part (1). Observe that

$$\frac{1}{x} \leq \frac{1}{n}$$

on the interval $[n, n+1]$. This implies that

$$\int_n^{n+1} \frac{1}{x}\,dx \leq \int_n^{n+1} \frac{1}{n}\,dx = \frac{1}{n}.$$

Therefore,

$$\frac{1}{n} - \int_n^{n+1} \frac{1}{x}\,dx \geq \frac{1}{n} - \frac{1}{n} = 0.$$

As for the upper bound, we observe that

$$\frac{1}{x} \geq \frac{1}{n+1}$$

on the interval $[n, n+1]$. This implies that

$$\int_n^{n+1} \frac{1}{x}\,dx \geq \int_n^{n+1} \frac{1}{n+1}\,dx = \frac{1}{n+1}.$$

It follows that

$$\frac{1}{n} - \int_n^{n+1} \frac{1}{x}\,dx \leq \frac{1}{n} - \frac{1}{n+1}.$$

$\square$

Solution to Part (2). Part (1) implies that

$$\sum_{n=1}^{N-1} \frac{1}{n} - \int_n^{n+1} \frac{1}{x}\,dx \leq \sum_{n=1}^{N-1} \frac{1}{n} - \frac{1}{n+1}. \tag{2.1}$$

We re-write the left-hand side first:

$$\sum_{n=1}^{N-1} \frac{1}{n} - \int_n^{n+1} \frac{1}{x}\,dx = \sum_{n=1}^{N-1} \frac{1}{n} - \sum_{n=1}^{N-1} \int_n^{n+1} \frac{1}{x}\,dx = \sum_{n=1}^{N-1} \frac{1}{n} - \int_1^N \frac{1}{x}\,dx.$$

As for the right-hand side,

$$\frac{1}{n} - \frac{1}{n+1} = \frac{(n+1)-n}{n(n+1)} = \frac{1}{n^2+n},$$

and so

$$\sum_{n=1}^{N-1} \frac{1}{n} - \frac{1}{n+1} = \sum_{n=1}^{N-1} \frac{1}{n^2+n}.$$

It follows from (2.1) that

$$\sum_{n=1}^{N-1} \frac{1}{n} - \int_1^N \frac{1}{x}\,dx \leq \sum_{n=1}^{N-1} \frac{1}{n^2+n}. \tag{2.2}$$

By adding $\frac{1}{N}$ to both sides, we obtain the desired inequality:

$$\sum_{n=1}^{N} \frac{1}{n} - \int_1^N \frac{1}{x}\,dx \leq \frac{1}{N} + \sum_{n=1}^{N-1} \frac{1}{n^2+n}.$$

□

Solution to Part (3). Since

$$\sum_{n=1}^{\infty} \frac{1}{n^2+n} = \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1},$$

we use the technique of telescoping sums to show that the series converges.

To this end, we set

$$S_N = \sum_{n=1}^{N} \frac{1}{n} - \frac{1}{n+1}$$

for each $N \in \mathbb{N}$. Note that

$$\begin{aligned} S_N &= \sum_{n=1}^{N} \frac{1}{n} - \sum_{n=1}^{N} \frac{1}{n+1} = \sum_{n=1}^{N} \frac{1}{n} - \sum_{n=2}^{N+1} \frac{1}{n} \\ &= 1 - \left(\sum_{n=2}^{N} \frac{1}{n} - \sum_{n=2}^{N} \frac{1}{n} \right) - \frac{1}{N+1} = 1 - \frac{1}{N+1}. \end{aligned}$$

It follows that

$$\sum_{n=1}^{\infty} \frac{1}{n^2+n} = \lim_{N\to\infty} s_N = \lim_{N\to\infty} 1 - \frac{1}{N+1} = 1,$$

whence

$$\sum_{n=1}^{\infty} \frac{1}{n^2+n}$$

converges (to 1). □

Solution to Part (4). Part (1) implies that

$$0 \leq \frac{1}{n} - \int_n^{n+1} \frac{1}{x}\, dx \leq \frac{1}{n^2+n}$$

for all $n \geq 1$, and Part (3) states that

$$\sum_{n=1}^{\infty} \frac{1}{n^2+n}$$

is convergent. It follows from the comparison test that the series

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \int_n^{n+1} \frac{1}{x}\, dx \right)$$

converges.

Now,

$$\begin{aligned} \sum_{n=1}^{\infty} \left(\frac{1}{n} - \int_n^{n+1} \frac{1}{x}\, dx \right) &= \lim_{N\to\infty} \sum_{n=1}^{N-1} \left(\frac{1}{n} - \int_n^{n+1} \frac{1}{x}\, dx \right) \\ &= \lim_{N\to\infty} \sum_{n=1}^{N-1} \frac{1}{n} - \int_1^{N} \frac{1}{x}\, dx. \end{aligned}$$

Since

$$\lim_{N\to\infty} \frac{1}{N} = 0,$$

the algebraic limit theorem shows that the limit

$$\lim_{N\to\infty} \frac{1}{N} + \left(\sum_{n=1}^{N-1} \frac{1}{n} - \int_1^{N} \frac{1}{x}\, dx \right)$$

exists. The computations carried out in Part (2) now implies that the above limit equals

$$\lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{n} - \int_1^{N} \frac{1}{x}\, dx.$$

This completes the proof. □

Problem 3 (10 points)**.** Use the power series for $\arctan x$ to show that

$$\pi = 2\sqrt{3} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n}.$$

Solution. Recall from class that

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1},$$

which is valid for $-1 < x < 1$. (Therefore, we cannot compute $\arctan 1$ or $\arctan \sqrt{3}$ with this power series expansion.) Since

$$\arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6},$$

we see that

$$\frac{\pi}{6} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \left(\frac{1}{\sqrt{3}}\right)^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n} \cdot \frac{1}{\sqrt{3}}.$$

It now follows that

$$\pi = \frac{6}{\sqrt{3}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n} = 2\sqrt{3} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n},$$

as was to be shown. □

Problem 4 (20 points)**.** Find the radius of convergence of the *Bessel function of order r*

$$J_r(x) = \left(\frac{x}{2}\right)^r \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+r)!} \left(\frac{x}{2}\right)^{2n},$$

where r is a positive integer. Show that $y = J_r(x)$ is a solution to *Bessel's differential equation*

$$x^2 y'' + xy' + (x^2 - r^2)y = 0.$$

Solution. The $\left(\frac{x}{2}\right)^r$ term is irrelevant in computing the radius of convergence: it suffices to investigate

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+r)!} \left(\frac{x}{2}\right)^{2n}.$$

We set

$$b_n = \frac{(-1)^n}{n!(n+r)!} \left(\frac{x}{2}\right)^{2n}$$

for each $n \geq 0$ and observe that

$$\begin{aligned} \left|\frac{b_{n+1}}{b_n}\right| &= \left|\frac{(-1)^{n+1} x^{2n+2}/2^{2+2n}(n+1)!(n+r+1)!}{(-1)^n x^{2n}/2^{2n} n!(n+r)!}\right| \\ &= \left|\frac{-x^2}{4(n+1)(n+r+1)}\right| = \frac{|x|^2}{4(n+1)(n+r+1)}. \end{aligned}$$

Since

$$\lim_{n\to\infty} \left|\frac{b_{n+1}}{b_n}\right| = 0 < 1$$

regardless of our choice of x, the ratio test implies that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+r)!} \left(\frac{x}{2}\right)^{2n}$$

converges absolutely on $(-\infty, \infty)$. It follows that the radius of convergence of J_r is ∞.

Now,

$$J_r(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+r)!} \left(\frac{x}{2}\right)^{2n+r},$$

and $r \geq 1$, and so J_r has no constant term. Therefore,

$$J_r'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2n+r)}{2(n!)(n+r)!} \left(\frac{x}{2}\right)^{2n+r-1}.$$

If $r = 1$, then

$$J_1'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n(2n+1)}{2(n!)(n+1)!} \left(\frac{x}{2}\right)^{2n},$$

and so $J_1'(x)$ has a constant term. Therefore,

$$J_1''(x) = \sum_{n=1}^{\infty} \frac{(-1)^n(2n+1)(2n)}{4(n!)(n+1)!} \left(\frac{x}{2}\right)^{2n-1}$$

If $r > 1$, then

$$J_r'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n(2n+r)}{2(n!)(n+r)!} \left(\frac{x}{2}\right)^{2n+r-1}$$

does not have a constant term, and so

$$J_r''(x) = \sum_{n=0}^{\infty} \frac{(-1)^n(2n+r)(2n+r-1)}{4(n!)(n+r)!} \left(\frac{x}{2}\right)^{2n+r-2}.$$

Note, however, that $r = 1$ implies that

$$\frac{(-1)^n(2n+r)(2n+r-1)}{4(n!)(n+r)!} \left(\frac{x}{2}\right)^{2n+r-2} = \frac{(-1)^n(2n+1)(2n)}{4(n!)(n+1)!} \left(\frac{x}{2}\right)^{2n-1}.$$

When $n = 0$, we obtain the following:

$$\frac{(-1)^n(2\cdot 0+1)(2\cdot 0)}{4(0!)(0+1)!} \left(\frac{x}{2}\right)^{2\cdot 0-1} = 0.$$

It follows that

$$J_r''(x) = \sum_{n=0}^{\infty} \frac{(-1)^n(2n+r)(2n+r-1)}{4(n!)(n+r)!} \left(\frac{x}{2}\right)^{2n+r-2}$$

in the $r = 1$ case as well, despite the presence of the $\frac{1}{x}$ term. In other words, there is no need to consider the $r = 1$ case and the $r > 1$ case separately.

Let us now verify that $J_r(x)$ satisfies Bessel's differential equation

$$x^2 J_r''(x) + xJ_r'(x) + (x^2 - r^2)J_r(x) = 0.$$

Since

$$x^2 J_r''(x) = \sum_{n=0}^{\infty} \frac{(-1)^n(2n+r)(2n+r-1)}{n!(n+r)!} \left(\frac{x}{2}\right)^{2n+r}$$

$$xJ_r'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n(2n+r)}{n!(n+r)!} \left(\frac{x}{2}\right)^{2n+r}$$

$$(x^2 - r^2)J_r(x) = \sum_{n=0}^{\infty} \frac{(-1)^n(x^2-r^2)}{n!(n+r)!} \left(\frac{x}{2}\right)^{2n+r},$$

we see that

$$
\begin{aligned}
&x^2 J_r''(x) + xJ_r'(x) + (x^2 - r^2)J_r(x) \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+r)!} \left(\frac{x}{2}\right)^{2n+r} \left((2n+r)(2n+r-1) + (2n+r) + (x^2 - r^2)\right) \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+r)!} \left(\frac{x}{2}\right)^{2n+r} \left(4n^2 + 4nr + x^2\right) \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n 4n(n+r)}{n!(n+r)!} \left(\frac{x}{2}\right)^{2n+r} + \sum_{n=0}^{\infty} \frac{(-1)^n x^2}{n!(n+r)!} \left(\frac{x}{2}\right)^{2n+r}.
\end{aligned}
$$

Now, when $n = 0$,

$$
\frac{(-1)^n 4n(n+r)}{n!(n+r)!} \left(\frac{x}{2}\right)^{2n+r} = \frac{(-1)^0 4 \cdot 0 \cdot (0+r)}{0!(0+r)!} \left(\frac{x}{2}\right)^{2\cdot 0+r},
$$

and so

$$
\sum_{n=0}^{\infty} \frac{(-1)^n 4n(n+r)}{n!(n+r)!} \left(\frac{x}{2}\right)^{2n+r} = \sum_{n=1}^{\infty} \frac{(-1)^n 4n(n+r)}{n!(n+r)!} \left(\frac{x}{2}\right)^{2n+r}.
$$

It now follows that

$$
\begin{aligned}
&x^2 J_r''(x) + xJ_r'(x) + (x^2 - r^2)J_r(x) \\
&= \sum_{n=1}^{\infty} \frac{(-1)^n 4n(n+r)}{n!(n+r)!} \left(\frac{x}{2}\right)^{2n+r} + \sum_{n=0}^{\infty} \frac{(-1)^n x^2}{n!(n+r)!} \left(\frac{x}{2}\right)^{2n+r} \\
&= \sum_{n=1}^{\infty} \frac{(-1)^n 4}{(n-1)!(n+r-1)!} \left(\frac{x}{2}\right)^{2n+r} + \sum_{n=0}^{\infty} \frac{(-1)^n 4}{n!(n+r)!} \left(\frac{x}{2}\right)^{2n+r+2} \\
&= \sum_{n=1}^{\infty} \frac{(-1)^n 4}{(n-1)!(n+r-1)!} \left(\frac{x}{2}\right)^{2n+r} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 4}{(n-1)!(n+r-1)!} \left(\frac{x}{2}\right)^{2n+r} = 0,
\end{aligned}
$$

as was to be shown. □