CALCULUS II, SUMMER 2015 - WEEKEND PROBLEM SET 5

60 points total = 50 points + 10 extra credit points

Name: _____ Score: _____/ 50

Use this worksheet as the cover sheet for your write-up: write your name on this page, and staple this sheet to the front of your homework packet.

Please indicate clearly which problems you have worked on. You will receive no credit for submitting solutions that the grader cannot read and understand—be sure to write legibly!

Problem 1 (10 points)**.** For each integer $n \geq 0$, we let

$$a_{2n} = a_{2n+1} = \frac{1}{2^n}.$$

Apply the ratio test and the root test on the series

$$\sum_{n=0}^{\infty} a_n.$$

What are the results?

Problem 2 (20 points)**.** Solve the following problems:

(1) For each integer $n \geq 1$, show that

$$0 \leq \frac{1}{n} - \int_n^{n+1} \frac{1}{x}\, dx \leq \frac{1}{n} - \frac{1}{n+1}.$$

(2) Use (1) to show that

$$\sum_{n=1}^{N} \frac{1}{n} - \int_1^N \frac{1}{x}\, dx \leq \frac{1}{N} + \sum_{n=1}^{N-1} \frac{1}{n^2+n}$$

for each $N \geq 2$.

(3) Show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2+n}$$

converges.

(4) Use (1), (2), and (3) to show that

$$\lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{n} - \int_1^N \frac{1}{x}\, dx$$

converges. The limit is called the *Euler–Mascheroni constant* and is often denoted by γ.

Date: August 10, 2015.

Problem 3 (10 points)**.** Use the power series for $\arctan x$ to show that

$$\pi = 2\sqrt{3}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)3^n}.$$

Problem 4 (20 points)**.** Find the radius of convergence of the *Bessel function of order r*

$$J_r(x) = \left(\frac{x}{2}\right)^r \sum_{n=0}^{\infty}\frac{(-1)^n}{n!(n+r)!}\left(\frac{x}{2}\right)^{2n},$$

where r is a positive integer. Show that $y = J_r(x)$ is a solution to *Bessel's differential equation*

$$x^2y'' + xy' + (x^2 - r^2)y = 0.$$