Notes from the 2015 Borel Seminar on High-Dimensional Expanders

Lectures by Jacob Fox, Tali Kaufman, Roy Meshulam, Janos Pach, Ori Parzanchevski, Benny Sudakov, and Uli Wagner

Notes by Mark Kim

The following set of notes consists of live-TeXed transcriptions of mini-courses given at the 2015 Borel Seminar on high-dimensional expanders. The conference took place at Les Diablerets, Switzerland from March 9, 2015 to March 13, 2015. For more information on the conference, please visit

http://borel2015.unige.ch.

The notes were produced real-time during each lecture. Moreover, not all talks are represented here. I stress that the transcriptions are not meant to be authoritative reproductions of the talks, and that all the errors herein are my own. Corrections and comments are gratefully received at

markhkim@math.nyu.edu.

The present document is up to date as of March 12, 2015; the current version of these notes can be found at

http://markhkim.com/writings

Mark H. Kim
Courant Institute, NYU

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CHAPTER 1

Jacob Fox: Regularity Lemmas, High-Dimensional Geometric Expanders

1. Regularity Lemmas (Day 1, Period 4)

DEFINITION 1.1. A *graph* $G = (V, E)$ has a vertex set V and a set E of edges, which are pairs of vertices.

$e(X, Y)$ denotes the number of pairs in $X \times Y$ that are edges.

THEOREM 1.2 (Expander mixing lemma). *If $G = (V, E)$ is a k-regular graph of size n with second largest (in absolute value) eigenvalue λ, then, for all $X, Y \subseteq$, we have the following inequality:*

$$\left| e(X, Y) - \frac{k}{n}|X||Y| \right| \leq \lambda\sqrt{|X||Y|}.$$

Roughly, Szemerédi's regularity lemma states that every large graph can be partitioned into a bounded number of roughly equally-sized parts so that the graph is random-like between almost all pairs of parts. This is considered one of the most powerful results in graph theory.

To state the result precisely, we must understand what "random-like" means. To this end, we let X and Y be vertex subsets of a graph G.

DEFINITION 1.3. The *density* of the pair (X, Y) is the quantity

$$d(X, Y) = \frac{e(X, Y)}{|X||Y|}.$$

DEFINITION 1.4. The *irregularity* $\operatorname{irreg}(X, Y)$ of the pair (X, Y) is the maximum, over all $A \subseteq X$ and $B \subseteq Y$ of the following quantity

$$|e(A, B) - d(X, Y)|A||B||\,.$$

We say that (X, Y) is ε*-regular* if $\operatorname{irreg}(X, Y) \leq \varepsilon|X||Y|$.

DEFINITION 1.5. The *irregularity* of a vertex partition P of a graph $G = (V, E)$ is

$$\operatorname{irreg}(P) = \sum_{X, Y \in P} \operatorname{irreg}(X, Y).$$

Partition P is said to be ε*-regular* if $\operatorname{irreg}(P) \leq \varepsilon|V|^2$.

The following is a statement equivalent to the original Szmerédi regularity lemma, due to Lovász and Szegedy:

THEOREM 1.6 (Szemerédi's regularity lemma). *For every $\varepsilon > 0$, there is an $M(\varepsilon)$ so that every graph G has an ε-vertex partition P with at most $M(\varepsilon)$ parts.*

A natural question to ask at this stage is: how big is $M(\varepsilon)$? *Tower function* $T(n)$ is given by $T(1) = 2$ and $T(n) = 2^{T(n-1)}$. Szemerédi's original proof gives a Tower-function upper bound. Moreover, Gowers proved that $M(\varepsilon) \geq T(\varepsilon^{-c})$ for some constant $c > 0$.

Gowers asked in 1997 what the precise order of the tower height of $M(\varepsilon)$. This was resolved by Fox, et al. in 2014:

THEOREM 1.7 (Fox-L. M. Lovász 2014). *$M(\varepsilon) = T(\Theta(\varepsilon^{-2}))$.*

Let us now sketch the proof of the regularity lemma. To this end, we introduce the following notion:

DEFINITION 1.8. For a vertex partition $P : V = V_1 \cup \cdots \cup V_k$, the *mean-square density* of P is declared to be the quantity

$$q(P) = \sum_{i,j} p_i p_j d^2(V_i, V_j),$$

where $p_i = \frac{|V_i|}{|V|}$.

We remark that $0 \leq q(P) \leq 1$ for all P. We also remark that if P' is a refinement of P, then $q(P') \geq q(P)$.

The key step is as follows:

CLAIM. *If P with $|P| = k$ is not ε-regular, then there is a refinement P' into at most $k2^{k+1}$ parts such that $q(P') \geq q(P) + \varepsilon^2$.*

At most ε^{-2} iterations are needed before obtaining an ε-regular partition, and this gives an upper bound. The current available proofs for lower bounds reverse-engineer this proof to obtain a construction.

There are many applications of the regularity lemma. The so-called *regularity method* is useful:

(1) apply szemeredi's regularity lemma

use a counting lemma for embedding small graphs

THEOREM 1.9 (Triangle counting lemma). *If each pair of parts is ε-regular, the number of triangles across the three parts is $\approx d(X,Y)d(X,Z)d(Y,Z)|X||Y||Z|$.*

Some applications are given below.

THEOREM 1.10 (Graph removal lemma). *For each $\varepsilon > 0$ and graph H on h vertices there is $\delta > 0$ such that every graph on n vertices with at most δn^h copies of H can be made H-free by removing εn^2 edges.*

SKETCH OF PROOF. Apply Szemerédi's regularity lemma, and delete edges between pairs which are irregular or sparse. If there is a remaining copy of H, then its edges go between pairs which are both dense and regular. The counting lemma then implies that there are at least δn^h copies of H. □

This was proved for triangles by Ruzsa and Szemerédi in 1976, and in general by Alon, Duke, and Lefmann, Rődl, and Yuster in 1994.

This has many applications in extremal graph theory, additive number theory, and theoretical computer science, and discrete geometry. For example, this implies Roth's theorem, which is the $k = 3$ case of the Szemerédi's theorem.

PROBLEM 1.11 (Erdős, Alon, Gowers, Tao). *Find a new proof which gives better bounds.*

THEOREM 1.12 (Fox). *δ^{-1} can be taken to be a tower of twos of height*

$$c_H \log \varepsilon^{-1}.$$

The best known lower bound on δ^{-1} is $\varepsilon^{-c\log\varepsilon^{-1}}$, due to Bernhard.

The key lemma for the new proof is as follows:

LEMMA 1.13. *If there are at most $\alpha|V_1||V_2||V_3|$ triangles across V_1, V_2, V_3, then there are $1 \leq i \leq j \leq 3$ and equitable partitions Q_i of V_i and Q_j of V_j each with at most $2^{\alpha^{-O(1)}}$ parts such that there are at least $\frac{1}{10}|Q_i||Q_j|$ pairs $(X, Y) \in Q_i \times Q_j$ with $d(X, Y) < 2\alpha^{1/3}$.*

DEFINITION 1.14. For a vertex partition $P : V = V_1 \cup \cdots \cup V_k$, the *mean entropy density* of P is the quantity

$$q(P) = \sum_{i,j} p_i p_j d(V_i, V_j) \log d(V_i, V_j),$$

where $p_i = \frac{|V_i|}{|V|}$.

Let us now discuss weak regularity lemmas. These give better bounds and are thus more useful for algorithmic applications. Nevertheless, these are not strong enough for the most interesting applications of the regularity lemma.

Here is an important example of a weak regularity lemma.

THEOREM 1.15 (Frieze–Kannan, 1999). *For $\varepsilon > 0$, there is $h = h(\varepsilon)$ such that every graph $G = (V, E)$ has an equitable partition $P : V = V_1 \cup \cdots \cup V_h$ such that for any subsets $S, T \subseteq V$,*

$$\left| e(S,T) \sum_{i,j} |S \cap V_j||T \cap V_j| d(V_i, V_j) \right| < \varepsilon |V|^2.$$

This is a fundamental algorithmic tool with only a single exponential bound. Unfortunately, the bound cannot be improved:

THEOREM 1.16 (Conlon–Fox). *$h(\varepsilon) = 2^{\Theta(\varepsilon^{-2})}$.*

Once again, the proof of this result mimics the Friesz–Kannan regularity lemma by reverse-engineering it.

For an algorithmic graph theory, we consider the problem of counting cliques. To this end, we let G be a graph with n vertices. Unless P = NP, there is no polynomial-time algorithm that approximates the maximum clique to within a factor better than $O(n^{1-\varepsilon})$ for any $\varepsilon > 0$.

But what about counting the number of K-cliques? The trivial algorithm is in time $O(n^k)$. The best known bound is $O(n^{\omega k/3})$, where $\omega \approx 2.37\ldots$ is the matrix multiplication constant.

THEOREM 1.17 (Duke–Lefmann–Rődl). *In a graph G on n vertices, we can approximate the count of cliques of order k within an additive εn^k in time $2^{O(k^2/\varepsilon^5)} n^2$.*

THEOREM 1.18 (Fox–Lovász–Zhao). *In a graph G on n vertices, we can approximate the count of cliques of order k within an additive εn^k in time $\varepsilon^{\binom{k}{2}} n - \varepsilon^{O(1)} n^\omega$.*

COROLLARY 1.19 (Fox–Lovász–Zhao). *In a graph G on n vertices, we can approximate the count of cliques of of order* 1000 *within an additive* $n^{10000-10^{-6}}$ *in time* $n^{2.4}$

The original regularity method is only useful for *dense graph*, but most practical problems concern sparse graphs. Toward this direction, Kohayakawa, Rődl, and Scott proved a sparse regularity lemma by mimicking the proof of the usual regularity lemma, making sure that there is no dense spot between any pair.

The problem, however, is that this does not give us a counting lemma in sparse graphs. In fact, a general counting lemma in sparse graphs cannot hold. How about within subgraphs of sparse pseudorandom graphs?

THEOREM 1.20 (Conlon–Fox–Zhao). *(A sparse counting lemma in graphs and hypergraphs satisfying some nice pseudorandomness condition.)*

An application of this result is a better relative Szemerédi theorem and a simpler proof of the Green–Tao theorem on long arithmetic progressions in primes.

We conclude this section by discussing hypergraph regularity lemmas.

DEFINITION 1.21. A *hypergraph* $H = (V, E)$ has a vertex set V and a set E of edges, which are subsets of V. It is *r-uniform* if each edge has size r.

Let X, Y, Z be vertex subsets of a 3-uniform hypergraph H. $e(X, Y, Z)$ is the number of triples in $X \times Y \times Z$ which are edges.

DEFINITION 1.22. The *density* of the triple (X, Y, Z) is the quantity.

$$d(X, Y, Z) = \frac{e(X, Y, Z)}{|X||Y||Z|}$$

DEFINITION 1.23. The *irregularity* $\operatorname{irreg}(X, Y, Z)$ of the triple (X, Y, Z) is the maximum, over all $A \subseteq X$, $B \subseteq Y$, and $C \subseteq Z$,

$$|e(A, B, C) - d(X, Y, Z)|A||B||C|| \,.$$

DEFINITION 1.24. The *irregularity* of a vertex partition P of a 3-uniform hypergraph $H = (V, E)$ is

$$\operatorname{irreg}(P) = \sum_{X, Y, Z \in P} \operatorname{irreg}(X, Y, Z).$$

Partition P is *ε-regular* if $\operatorname{irreg}(P) \le \varepsilon |V|^3$.

THEOREM 1.25 (Chung's hyergraph regularity lemma). *For every $\varepsilon > 0$, there is an $M(\varepsilon)$ so that every graph G has an ε-regular vertex partition P with at most $M(\varepsilon)$ parts.*

The problem with this result is that it can only be used to count linear hypergraphs.

A new hypergraph regularity method was developed independently by Nagle–Rődl–Schacht–Skokan, Gowers, and Tao.

DEFINITION 1.26. For a graph G and 3-uniform hypergraph H, we let $e_G(H)$ be the number of triangles in G which are edges of H. We also let $K_3(G)$ denote the number of triangles in G. Finally, we let $d_G(H)$ denote the quantity $e_G(H)/K_3(G)$, the edge density of H with respect to G.

A better definition of irregularity in this context is as follows:

DEFINITION 1.27. The *irregularity* $\mathrm{irreg}_G(H)$ of H is the maximum, over all subgraphs $G' \subseteq G$, of the quantity

$$|e_{G'}(H) - d_G(H)|K_3(G)||\,.$$

The hypergraph regularity lemma gives a vertex partition. Not only that, between each pair of parts, we also have a partition of complete bipartite graph in between there, and each of those is very regular.

2. Overlapping Simplices and High-Dimensional Expanders (Day 3, Period 2)

THEOREM 2.1 (Boros–Füredi). *For any n points in $\mathbb{R}^2$, there is a point $p \in \mathbb{R}^2$ in*

$$\left(\frac{2}{9} + o(1)\right)\binom{n}{3}$$

triangles induced by the n points.

Bukh gave a nice proof of the above theorem, using the following lemma:

LEMMA 2.2 (Ceder). *For any n points in $\mathbb{R}^2$, there are three concurrent lines that divide $\mathbb{R}^2$ into six parts each containing at least $\frac{n}{6} - 1$ points.*

THEOREM 2.3 (Bárány). *For any n points in $\mathbb{R}^d$, there is a point $p \in \mathbb{R}^d$ in at least*

$$(c(d) + o(1))\binom{n}{d+1}$$

simplices induced by the n points.

The current best lower bound is due to Gromov; the current best upper bound is due to Bukh–Matoušek–Nivasch.

$$\frac{2d}{(d+1)!(d+1)} \le c(d) \le \frac{d!}{(d+1)^d}$$

THEOREM 2.4 (Pach's selection theorem). *For any n points in $\mathbb{R}^d$, there are disjoint subsets $P_1, \ldots, P_{d+1}$ each of of size at least $c'_d n$ and a point $p \in \mathbb{R}^d$ such that every simplex with one point in each P_i contains p.*

See Chapter 6, Theorem 1.24 for details on the selection theorem.

DEFINITION 2.5. For a $(d+1)$-uniform hypergraph H, its *overlap number* $c(H)$ is the largest $C \in (0,1]$ such that for every embedding $f : V \to \mathbb{R}^d$, there exists a point $p \in \mathbb{R}^d$ which belongs to at least $c|E|$ simplices whose vertex sets are hyperedges of H.

DEFINITION 2.6. An infinite family $\mathcal{H}$ of $(d+1)$-uniform hypergraphs is *highly overlapping* if there exists an absolute constant $c > 0$ such that $c(H) > c$ for every $H \in \mathcal{H}$.

Gromov asked the following question:

PROBLEM 2.7 (Gromov). *There is a highly overlapping infinite family of $(d+1)$-uniform hypergraphs of bounded degree without isolated vertices.*

To this end, there are the following results:

THEOREM 2.8 (Fox–Gromov–Lafforgue–Naor–Pach). *For each d and $\varepsilon > 0$, there is $k = k(\varepsilon, d)$ and an infinite family of k-regular $(d+1)$-uniform hypergraphs H with $c(H) > c(d) - \varepsilon$.*

THEOREM 2.9 (Fox–Gromov–Lafforgue–Naor–Pach). *For all d, Δ, and $\varepsilon > 0$, there is n_0 such that every $(d+1)$-uniform hypergraph H on $n \geq n_0$ nonisolated vertices with maximum degree Δ satisfies $c(H) \leq c(d) + \varepsilon$.*

Let us now sketch the proof.

DEFINITION 2.10. A $(d+1)$-uniform hypergraph $H = (V, E)$ is *α-uniform* if, for all vertex subsets $X_1, \ldots, X_{d+1}$,

$$\left| e(X_1, \ldots, X_{d+1}) - \frac{|E|}{\binom{n}{d+1}} |X_1||X_2| \cdots |X_{d+1}| \right| \leq \alpha e(H).$$

We can either do an explicit construction via Ramanujan complexes, or just do a random construction via a first moment argument.

THEOREM 2.11 (Fox–Gromov–Lafforgue–Naor–Pach). *For each $\varepsilon > 0$, there is $\alpha > 0$ such that every $(d+1)$-uniform hypergraph $H = (V, E)$ which is α-uniform has $c(H) = c(d) \pm \varepsilon$.*

LEMMA 2.12 (Semi-algebraic hypergraph regularity lemma). *For each t, d, and $\varepsilon > 0$, there is K such that the vertex set of every $(d+1)$-uniform hypergraph X of complexity t has an equitable partition into K parts such that all but at most an ε-fraction of the $(d+1)$-tuples of parts are complete or empty.*

Here's a sketch of proof for the lower bound of Theorem 2.11. Let p be the point in the maximum number of $(d+1)$-simplices induced by points in V. Apply the semi-algebraic hypergraph regularity lemma to the hypergraph X_p on V whose edges are those $(d+1)$-tuples whose simplex contains p. Use α-uniformity for each complete $(d+1)$-tuple of parts.

CHAPTER 2

Tali Kaufman: High-Dimensional expanders and property testing

1. One-Dimensional Expanders (Day 2, Period 5)

We are interested in the following question:

> Let M be an n-by-n matrix whose entries are ± 1 and diagonal entries are 1. Could we query M in *constant* many locations and:
> - say `yes` if M is a tensor power $M = v \cdot v^t$ for some vector v;
> - say `no` with probability at least ε if M is ε-*far* from being a tensor power, viz., if we need to change at least εn^2 entries of M to get a tensor power.

This question arises in the field of *property testing*.

Definition 1.1. A graph $X = (V, E)$ is said to be an ε-*expander* if the *Cheeger constant*

$$h(X) = \min_{\substack{A \subsetneq V \\ A \neq \varnothing}} \frac{|E(A, V \smallsetminus A)|}{\min(|A|, |V \smallsetminus A|)}$$

is bounded below by ε.

Example 1.2. K_n, the complete graph of order n, is an expander.

Given a finite set A, a *property* is a subset $P \subset A^n$.

Example 1.3. Let $A = \mathbb{F}_2$ and P be the set of all $f : \mathbb{F}_2^n \to \mathbb{F}_2$ such that, for all $(x, y) \in \mathbb{F}_2^n$,

$$f(x) + f(y) + f(x + y) = 0.$$

P is (q, ε)-*testable* if there exists an algorithm that, given $f \in A^n$, reads q entries of f and outputs

- `yes` if $f \in P$ with probability 1;
- `no` if $f \notin P$ with probability at least $\varepsilon \operatorname{dist}(f, P)$.

Here $\operatorname{dist}(f, P)$ is the fraction of entries needed to be changed in f to have P.

Theorem 1.4 (BLR). *Linear functions are $(3, \frac{2}{9})$-testable. For the tester, choose $x, y \in \mathbb{F}_2^n$ and check $f(x) + f(y) + f(x + y) \geq 0$.*

Given a graph $X = (V, E)$, we define $P_V(X)$, the set of vertices that do not expand, i.e.,

$$\{A : V \to \{0, 1\} : \forall (u, v) \in E \ A(u) + A(v) = 0 \bmod 2\}.$$

X is an ε-expander if and only if $P_V(X) = \{\varnothing, V\} = \texttt{CONST}$ and $P_V(X)$ is $(2, \varepsilon)$-testable by its canonical test.

In light of this connection, we define a two-dimensional expander as follows.

Definition 1.5. Let $X = (V, E, T)$ be a 2-dimensional simplicial complex. Define $P_V(X)$ to be the set of vertices that do not expand. Define $P_E(X)$ to be the set of edges that do not expand, i.e.,

$$P_E(X) = \{f : E \to \{0, 1\} : \forall (e_1, e_2, e_3) \in T\ f(e_1) + f(e_2) + f(e_3) = 0\}.$$

We call the mandatory sets of edges that never expands `CUTS`.

X is an *ε-expander* if we have vertex expansion and edge expansion, viz.,

1. $P_V(X) = \texttt{CONST}$;
2. $P_V(X)$ is $(2, \varepsilon)$-testable by its canonical test;
3. $P_E(X) = \texttt{CUTS}$;
4. $P_E(X)$ is $(3, \varepsilon)$-testable by its canonical test.

CHAPTER 3

Alex Lubotzky: From Ramanujan Graphs to Ramanujan Complexes

1. Spectral Theory of Graphs (Day 1, Period 1)

Let $X = (V, E)$ be a graph. We shall assume that X is finite and *k-regular*, i.e., the degree of each vertex is k. Let $n = |V|$. Define the *adjacency matrix of* X to be the matrix $A = (A_{uv})$, where A_{uv} is the number of edges between u and v. We often think of A as a linear operator from $L^2(V)$ to $L^2(V)$, in the following manner:

$$(Af)(v) = \sum_{w \sim v} f(w).$$

We establish the basic properties of the adjacency matrix below; to this end, we recall that the *spectrum* of an n-by-n symmetric matrix B is the set

$$\operatorname{spec}(B) = \{\mu_0 \geq \mu_1 \geq \cdots \geq \mu_{k-1}\},$$

and the *spectral radius* of B is

$$r(B) = \max_{\lambda \in \operatorname{spec}(B)} |\lambda|.$$

PROPOSITION 1.1. *Let A be the adjacency matrix of a k-regular graph $X = (V, E)$.*

(1) A is a symmetric matrix, and so every eigenvalue of A is real.
(2) The spectral radius of A is bounded by k.
(3) k is an eigenvalue of A and is of multiplicity 1 if and only if X is connected.
(4) $-k$ is an eigenvalue of A if and only if X is bipartite. Indeed, if X is bipartite, then the spectrum $\operatorname{spec}(A)$ of A is symmetric, viz., λ is an eigenvalue of A if and only if $-\lambda$ is an eigenvalue of A. (The notion of bipartite graphs is recalled below.)

EXAMPLE 1.2 (Bipartite graphs). Let $X = (V, E)$ be a graph. X is said to be *r-partite* for some fixed integer $r \geq 2$ in case there exists a partition of V into r sets $V_1, \ldots, V_n$ such that $xy \notin E$ whenever both x and y are in V_i for some i. In other words, all edges must connect vertices in different parts of the partition. If $r = 2$, we often use the term *bipartite* instead of *2-partite*.

PROOF OF PROPOSITION 1.1. (1) is just the spectral theorem.

(2) Let λ be an eigenvalue of A, and $f \in L^2(V)$ a corresponding nonzero eigenfunction. Since X is finite, there exists a $v_0 \in V$ such that $|f(v_0)| = \max_{v \in V} |f(v)|$. Observe now that

$$|\lambda f(v_0)| = |(Af)(v_0)| = \left| \sum_{w \sim v_0} f(w) \right| \leq \sum_{w \sim v_0} |f(w)| \leq k|f(v_0)|,$$

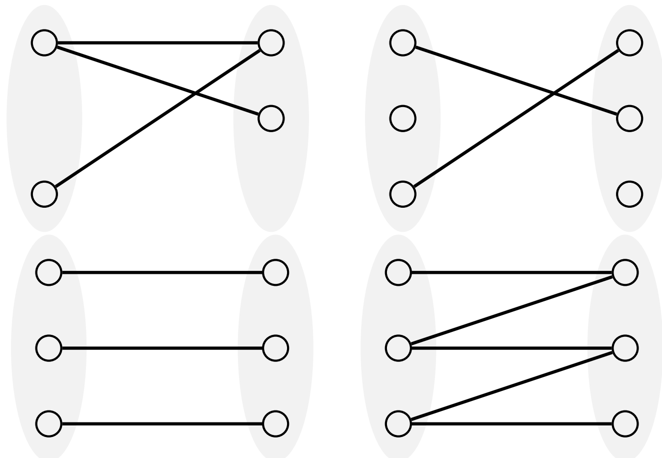


FIGURE 1. Four bipartite graphs

whence the desired result follows.

(3) The constant function $\mathbf{1} : V \to V$ given by the formula $\mathbf{1}(v) = v$ for all $v \in V$ is an eigenfunction for k, as X is k-regular.

Let f be a nonzero eigenfunction for k. Let $\{C_1, \ldots, C_l\}$ be the connected components of V. For each $1 \leq i \leq l$, we define

$$v_i = \max_{v \in C_i} |f(v)|.$$

Since

$$|kf(v_i)| = |(Af)(v_i)| = \left| \sum_{w \sim v_i} f(w) \right| \leq \sum_{w \sim v_i} |f(w)| \leq \sum_{w \sim v_i} |f(v_i)| = k|f(v_i)|,$$

we see that

$$\sum_{w \sim v_i} |f(w)| = \sum_{w \sim v_i} |f(v_i)|.$$

It then follows that $|f(w)| = |f(v_i)|$ for all $w \sim v_i$, and we can repeat this argument as needed to conclude that $|f(v)| = |f(v_i)|$ for all $v \in C_i$.

Now, we define

$$\tilde{v}_i = \max_{v \in C_i} f(v).$$

Since

$$kf(v_i) = (Af)(v_i) = \sum_{w \sim v_i} f(w) \leq \sum_{w \sim v_i} f(v_i) = kf(v_i),$$

we see that

$$\sum_{w \sim v_i} f(w) = \sum_{w \sim v_i} f(v_i).$$

We already know that $|f(v)| = |f(v_i)|$ for all $v \in C_i$, and so the above identity holds only if $f(w) = f(v_i)$ for all $w \sim v_i$. We now repeat this argument as needed to conclude that $f(v) = f(v_i)$ for all $v \in C_i$.

What we have shown is that every nonzero eigenfunction of A with respect to k must be *locally constant*, i.e., constant on each connected component. It follows that the multiplicity of k is precisely the number of the connected components of X.

(4) Suppose that X is bipartite with parts V_1 and V_2. Fix an eigenvalue λ of A and an eigenfunction $f \in l^2(X)$ corresponding to λ. The function

$$\bar{f}(v) = \begin{cases} f(v) & \text{if } v \in V_1; \\ -f(v) & \text{if } v \in V_2 \end{cases}$$

is an eigenfunction for $-\lambda$, i.e., $A\bar{f} = \lambda\bar{f}$. In particular, $-k$ is an eigenvalue of A.

Conversely, we suppose that $-k$ is an eigenvalue of A. This means that

$$\sum_{w \in v} f(w) = -kf(v)$$

for all $v \in V$. We let $V_1 = \{v \in V : f(v) = \|f\|_\infty\}$ and $V_2 = \{v \in V : f(v) = -\|f\|_\infty\}$.

For each $v \in V_1$, we have

$$-k\|f\|_\infty = -kf(v) = \sum_{w \sim v} f(w).$$

Since $f(w) \geq -\|f\|_\infty$ for all $w \sim v$, the above identity implies that $f(w) = -\|f\|_\infty$ for all $w \sim v$. In other words, if $v \in V_1$, then $w \in V_2$ for all $w \sim v$.

Similarly, we can show that $v \in V_2$ implies $w \in V_1$ for all $w \sim v$. □

Definition 1.3. Let X be a connected graph. We define

$$\lambda(X) = \max\left\{|\lambda| : \lambda \in \operatorname{spec}(A) \setminus \{-k, k\}\right\}.$$

$\lambda(X)$ controls the rate of convergence of the random walk on X to the uniform distribution. To see this, we let $\vec{p}$ be a probability distribution on V, viz., $0 \leq p_i = p(v_i) \leq 1$ for all i and $\sum_i p_i = 1$. If, in particular, $\vec{p}^{(0)} = (1, 0, \ldots, 0)$, then

$$\vec{p}^{(t+1)} = M\vec{p}^{(t)},$$

where $M = \frac{1}{k}A$.

Now, if $\alpha_0, \ldots, \alpha_{n-1}$ is an orthonormal basis of eigenfunctions of M corresponding to the eigenvalues

$$1 = \tilde{\mu}_0 \geq \tilde{\mu}_1 \geq \cdots \geq \tilde{\mu}_{n-1} \geq -1,$$

then we can write

$$\vec{p}^{(0)} = \sum_{i=0}^{n-1} a_i^{(0)} \alpha_i,$$

where $a_i^{(0)} = \langle \vec{p}^{(0)}, \alpha_i \rangle$. Since

$$\alpha_0 = (n^{-1/2}, \ldots, n^{-1/2}),$$

we see that

$$a_0^{(0)} = \langle \vec{p}^{(0)} \alpha_0 \rangle = \sum_{i=0}^{n-1} \frac{1}{\sqrt{n}} p_i^{(0)} = \frac{1}{\sqrt{n}}.$$

Observe now that

$$M^t(\vec{p}^{(0)}) = M^t\left(\sum_{i=0}^{n-1} a_i^{(0)} \alpha_i\right) = \sum_{i=0}^{n-1} a_i^{(0)} \tilde{\mu}_i^t \alpha_i,$$

and so

$$\lim_{t\to\infty} M^t(\vec{p}^{(0)}) = a_0^{(0)}\alpha_0 = (n^{-1}, \ldots, n^{-1}),$$

as was to be shown.

Unfortunately, $\lambda(X)$ is subject to the following restriction:

THEOREM 1.4 (Alon–Boppana). *If $\{X_r = (V_r, E_r)\}_{r=1}^{\infty}$ is a family of k-regular graphs such that $|X_r| \to \infty$ as $r \to \infty$, then*

$$\liminf_{r\to\infty} \lambda(X_r) \geq 2\sqrt{k-1}.$$

EXAMPLE 1.5. If X is a k-regular graph, then the k-regular infinite tree T_k is the universal cover of X with respect to the shortest-path metric topology on X. A result of Kesten shows that, if A is the adjacency matrix of T_k, then $r(A) = 2\sqrt{k-1}$. See, for example, the following blog post for a proof: https://lucatrevisan.wordpress.com/2014/08/20/the-spectrum-of-the-infinite-tree/

DEFINITION 1.6. A finite k-regular graph is said to be a *Ramanujan graph* if $\lambda(X) \leq 2\sqrt{k-1}$.

EXAMPLE 1.7. Ramanujan graphs exist. The construction of a standard example requires the classical theorem of Jacobi, which states that

$$r_4(n) = |\{(x_0, x_1, x_2, x_3) \in \mathbb{Z}^4 : x_0^2 + x_1^2 + x_2^2 + x_3^2 = n\}| = 8 \sum_{\substack{d|n \\ 4\nmid d}} d. \tag{1.8}$$

We fix two distinct primes p and q such that $p, q \equiv 1 \bmod 4$. By the Jacobi theorem, we see that $r_4(p) = 8(p+1)$. Note that if $p = x_0^2+x_1^2+x_2^2+x_3^2$, then one of the x_i's is odd and the other three are even. Let S be the set of $\alpha = (x_0, x_1, x_2, x_3)$ such that $x_0^2 + x_1^2 + x_2^2 + x_3^2 = p$, $x_0 > 0$ is odd, and x_1, x_2, and x_3 are even. Then $|S| = p + 1$.

Observe that there exists an $\varepsilon \in \mathbb{F}_q$ such that $\varepsilon^2 = -1$. For each $\alpha \in S$, we see that

$$\alpha' = \begin{pmatrix} x_0 + x_1\varepsilon & x_2 + x_3\varepsilon \\ -x_2 + x_3\varepsilon & x_0 - x_1\varepsilon \end{pmatrix}$$

is in $PGL_2(\mathbb{F}_q)$.

For each $\alpha \in S$, we see that the quaternion conjugate $\bar{\alpha}$ is an element of S. Moreover, $\bar{\alpha}' = (\alpha')^{-1}$. It now follows that

$$S' = \{\alpha' : \alpha \in S\}$$

is a symmetric subset of $PGL_2(\mathbb{F}_q)$.

Let $H = \langle S' \rangle$, the subgroup of $PGL_2(\mathbb{F}_q)$ generated by S'. Let $X^{p,q} = \mathrm{Cay}(H; S')$, viz., the graph (V, E) obtained by setting $V = H$ and declaring $a \sim as'$ for each $a \in H$ and every $s \in S'$. Then $X^{p,q}$ is a $(p+1)$-regular Ramanujan graph. Moreover, if the quadratic residue $(\frac{p}{q})$ equals 1, then $H = PSL_2(\mathbb{F}_q)$ and $X^{p,q}$ is non-bipartite. On the other hand, if $(\frac{p}{q}) = -1$, then $H = PGL_2(\mathbb{F}_q)$ and $X^{p,q}$ is bipartite.

In either case, $|\lambda(X^{p,q})| \leq 2\sqrt{p}$. The proof of this result is closely related to the Riemann Hypothesis over finite fields. $\square$

2. Arithmetic Groups (Day 2, Period 1)

Let G be a Lie group. A discrete subgroup Γ of G is called a *lattice* if G/Γ has a finite G-invariant measure.

EXAMPLE 2.1. Γ is lattice if G/Γ is compact, as compact groups always admit Haar measures. For example, take $G = \mathbb{R}^n$ and $\Gamma = \mathbb{Z}^n$.

EXAMPLE 2.2. $SL_n(\mathbb{R})$ is a Lie group, and $SL_n(\mathbb{Z})$ is a lattice, even though $SL_n(\mathbb{R})/SL_n(\mathbb{Z})$ is not compact.

Let $n = 2$ and take $G = SL_2(\mathbb{R})$. Each $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$ acts on the upper half-plane $U = \{x + iy : x \in \mathbb{R} \text{ and } y > 0\}$ by Möbius transformations

$$z \mapsto \frac{az + b}{cz + d}.$$

This is a transitive action, and $\operatorname{stab}(i) = SO(2)$, the maximal compact subgroup of G. We can therefore identify U with the coset space G/K. (Careful: this is not a quotient group!) G/K is commonly referred to as the *hyperbolic plane*.

If $\Gamma \leq G$ is a cocompact lattice, viz., the quotient G/Γ is compact, then $\Gamma\backslash U = \Gamma\backslash G/K$, a Riemann surface.

In lieu of giving a precise definition upfront, we start by considering some examples of arithmetic lattices.

EXAMPLE 2.3. Consider the map $f : \mathbb{Z}[\sqrt{2}] \to \mathbb{R} \times \mathbb{R}$ given by the formula

$$f(a + b\sqrt{2}) = (a + \sqrt{2}b, a - \sqrt{2}b).$$

$\operatorname{im} f$ is a discrete subgroup of $\mathbb{R} \times \mathbb{R}$.

Analogously, $SL_n(\mathbb{Z}[\sqrt{2}])$ can be thought of as a discrete subgroup of $SL_n(\mathbb{R}) \times SL_n(\mathbb{R})$.

EXAMPLE 2.4. For each quadratic form Q, we define

$$SO(Q, \mathbb{R}) = \{A \in GL_{n+1}(\mathbb{R}) : Q(Av) = Q(v)\}.$$

Consider the quadratic form

$$f(x_1, \dots, x_{n+1}) = x_1^2 + \cdots + x_n^2 - x_{n+1}^2.$$

Let us also take another quadratic form

$$h(x_1, \dots, x_{n+1}) = x_1^2 + \cdots + x_n^2 - \sqrt{2}x_{n+1}^2.$$

We shall consider the embedding $SO(h, \mathbb{Z}[\sqrt{2}]) \hookrightarrow SO(h, \mathbb{R}) \times SO(h^\tau, \mathbb{R})$. Here

$$h^\tau(x_1, \dots, x_{n+1}) = x_1^2 + \cdots + x_n^2 + \sqrt{2}x_{n+1}^2,$$

which is a quadratic form of signature $(n + 1, 0)$.

We have that

$$SO(h, \mathbb{R}) \times SO(h^\tau, \mathbb{R}) = SO(n, 1) \times SO(n + 1),$$

where $SO(n+1)$ is compact. Projecting to the first component, we obtain a lattice in $SO(n, 1)$. This is a consequence of the following general fact:

> If Γ is discrete in $G \times H$ and if H is compact, then Γ is discrete in G.

□

EXAMPLE 2.5. $\mathbb{Z}$ is neither discrete nor dense in $\mathbb{Q}_p$. If, however, we embed $\mathbb{Z}[\frac{1}{p}]$ diagonally into $\mathbb{R} \times \mathbb{Q}_p$, then it is discrete. Consider now the embedding $SL_n(\mathbb{Z}[\frac{1}{p}]) \hookrightarrow SL_n(\mathbb{R}) \times SL_n(\mathbb{Q}_p)$. We shall imitate the construction in Example 2.4 to obtain a lattice.

To this end, we let R be a commutative ring and consider the unit quaternion space

$$\mathbb{H}^1(R) = \{\alpha = x_0 + x_1 i + x_2 j + x_3 k : x_0, x_1, x_2, x_3 \in R \text{ and } \|\alpha\| = 1\},$$

which is a group with respect to multiplication.

We first consider the $R = \mathbb{R}$ case. $\mathbb{H}^1(\mathbb{R}) \cong \mathbb{S}^3 \cong SU(2)$, which is a compact group.

Let us now assume that p is a prime number that is congruent to 1 mod 4. In this case, we have $\mathbb{H}^1(\mathbb{Q}_p) \cong SL_2(\mathbb{Q}_p)$. To see this, we take $\varepsilon \in \mathbb{Q}_p$ with $\varepsilon^2 = -1$ and send

$$\alpha \mapsto \begin{pmatrix} x_0 + x_1\varepsilon & x_2 + x_3\varepsilon \\ -x_2 + x_3\varepsilon & x_0 - x_1\varepsilon \end{pmatrix}.$$

We now consider the embedding

$$\mathbb{H}^1\left(\mathbb{Z}\left[\frac{1}{p}\right]\right) \xhookrightarrow{\text{discrete}} \mathbb{H}^1(\mathbb{R}) \times \mathbb{H}^1(\mathbb{Q}_p).$$

Since

$$\mathbb{H}^1(\mathbb{R}) \times \mathbb{H}^1(\mathbb{Q}_p) = SU(2) \times SL_2(\mathbb{Q}_p),$$

we see that

$$\Gamma = \mathbb{H}^1\left(\mathbb{Z}\left[\frac{1}{p}\right]\right) \hookrightarrow SL_2(\mathbb{Q}_p)$$

is a discrete embedding. □

Now, we fix $q \neq p$ and let $\Gamma(2q)$ be the kernel of the map

$$\mathbb{H}^1\left(\mathbb{Z}\left[\frac{1}{p}\right]\right) \to \mathbb{H}^1\left(\mathbb{Z}\left[\frac{1}{p}\right]/2q\mathbb{Z}\left[\frac{1}{p}\right]\right). \tag{2.6}$$

$\Gamma(2q)$ is a subgroup of finite index. The 2 in the parameter $2q$ makes $\Gamma(2q)$ torsion-free. Now, recall $X^{p,q}$ from Example 1.7. Here is the connection: we have that $X^{p,q} = \Gamma(2q)\backslash G/K$, where $G = PGL_2(\mathbb{Q}_p)$ and $K = PGL_2(\mathbb{Z}_p)$. We show in the next section that G/K is a Bruhat–Tits tree.

3. Bruhat–Tits trees (Day 3, Period 1)

Recall that the *field of p-adic numbers* is the collection

$$\mathbb{Q}_p = \left\{\sum_{i=-m}^{\infty} a_i p^i : a \le a_i \le p-1 \text{ and } m \in \mathbb{Z}\right\}$$

and that the *ring of p-adic integers* is the collection

$$\mathbb{Z}_p = \left\{\sum_{i=0}^{\infty} a_i p^i : a \le a_i \le p-1\right\}.$$

We have the following inclusion relations:

$$\overline{\mathbb{Z}} \subseteq \mathbb{Z}_p \subseteq \mathbb{Q}_p$$

If $x \in \mathbb{Q}_p$, then $p^l \in \mathbb{Z}_p$ for some l. Let $V = \mathbb{Q}_p^2$, for every basis $\{\alpha, \beta\}$ of V, we write L to denote $\mathbb{Z}_p\alpha + \mathbb{Z}_p\beta$. This L is called a *lattice*. (The choice of the term "lattice" is an unfortunate coincidence, as L not discrete. Since "lattice" the standard term for this notion in the literature, we begrudgingly use it here as well.)

Two lattices L_1 and L_2 are *equivalent* if there exists a nonzero element μ of $\mathbb{Q}_p$ such that

$$\mu L_1 = L_2.$$

This gives rise to an equivalence relation, whence we can speak of the equivalence class $[L]$ of a lattice L.

We say that an equivalence class of lattices $[L_1]$ is *adjacent* to another equivalence class of lattices $[L_2]$ if there exist representatives $L_1' \in [L_1]$ and $L_2' \in [L_2]$ such that

$$pL_1' \subsetneq L_2' \subsetneq L_1'.$$

We let

$$L_0 = \mathbb{Z}_p \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mathbb{Z}_p \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and observe that every lattice $L = \mathbb{Z}_p\alpha + \mathbb{Z}_p\beta$ admits an exponent l such that

$$p^l L \subseteq L_0.$$

We also note that

$$L/pL = (\mathbb{Z}_p/p\mathbb{Z}_p)\alpha + (\mathbb{Z}_p/p\mathbb{Z}_p)\beta \cong \mathbb{F}_p^2.$$

Therefore, every $[L]$ has $(p+1)$ neighbors, i.e., equivalence classes that are adjacent to $[L]$.

Let us now recall that a *tree* is a graph such that between two vertices there is a unique path. Indeed, given $[L]$, there exists a unique l such that $L_0/p^l L$ is a cyclic p-group. We can then connect L_0 and $p^{-1}(p^l L)$ by going one step at a time.

$$\begin{array}{c} L_0 \\ | \\ L_1 = pL \\ | \\ \vdots \\ | \\ L_{p-1} = p^{-1}(p^l L) \end{array}$$

(For a formal proof, see J. P. Serre, *Trees*.)

We now set $V = \mathbb{Q}_p^2$. We claim that

$$\{[L] : L = \mathbb{Z}_p\alpha + \mathbb{Z}_p\beta, \text{ where } \{\alpha, \beta\} \text{ is a basis for } V\}$$

is the vertex set of the $(p+1)$-regular tree T_{p+1}. $\tilde{G} = GL_2(\mathbb{Q}_p)$ acts transitively on bases of V, and the center $Z = Z(GL_2(\mathbb{Q}_p))$, the subgroup of scalar matrices, preserves the equivalence relation. Therefore, $PGL_2(\mathbb{Q}_p) = \tilde{G}/Z$ acts on the vertices.

We set $G = PGL_2(\mathbb{Q}_p)$ and set $K = PGL_2(\mathbb{Z}_p)$ Observe that $K = \text{stab}([L_0])$ with respect to the action of G, and that K is a maximal compact subgroup of G. It follows that $G/K \cong T_{p+1}$.

We now recall the Jacobi theorem (1.8). Let

$$S = \{\alpha = x_0 + x_1 i + x_2 j + x_3 k : x_0 > 0, x_0 \text{ is odd}, x_1, x_2, x_3 \text{ are even.}\}.$$

Then $|S| = p + 1$ and $\varepsilon = \sqrt{-1} \in \mathbb{Q}_p$.

Take $\alpha \in S$. Then $\alpha \in \mathbb{H}(\mathbb{Z}[\frac{1}{p}])$, $\|\alpha\| = p$, and p is invertible in $\mathbb{Z}[\frac{1}{p}]$. It follows that α is invertible in $\mathbb{H}(\mathbb{Z}[\frac{1}{p}]ee,)$; indeed:

$$\alpha^{-1} = \frac{\bar{\alpha}}{\|\alpha\|}.$$

We also note that $\alpha \equiv 1 \bmod 2$.

We now set $\Gamma = \mathbb{H}^*(\mathbb{Z}[\frac{1}{p}])/Z$, so that $S \subseteq \Gamma(2)$; see (2.6) for the definition of $\Gamma(2)$.

Claim. $\Gamma(2) = \langle S \rangle$.

$\langle S \rangle$ acts simply transitively on the tree G/K, and so $\langle S \rangle \cdot K = G$ and $\langle S \rangle$ is a free group on S. It now follows from the above claim that

$$T_{p+1} \cong \text{Cay}(\Gamma(2); S),$$

whence

$$\Gamma(2q)\backslash T_{p+1} \cong \Gamma(2q)\backslash G/K = \text{Cay}\left(\Gamma(2q)\backslash\Gamma(2); S\right).$$

Now would be a good time to go back and read Example 1.7 again!

4. Representation Theory (Day 4, Period 1)

4.1. Ramanujan Graphs and Tempered Representations. We now translate the combinatorial problem of constructing Ramanujan graphs to a problem in representation theory.

Proposition 4.1. *Let Λ be a cocompact subgroup of G. $\Lambda\backslash G/K$ is Ramanujan if and only if every infinite-dimensional, irreducible, spherical subrepresentation of $L^2(\Lambda\backslash G)$ is tempered.*

This gives us the desired result, as the next theorem furnishes precisely the requisite conditions:

Theorem 4.2 (Deligne). *If $\Lambda = \Gamma(2q)$ is a congruent subgroup as before, then every infinite-dimensional, irreducible, spherical subrepresentation of $L^2(\Gamma(2q)\backslash G)$ is tempered.*

Let $C = C_c(K\backslash G//K)$ be the complex bi-K-invariant functions in G with compact support, i.e.,

$$f(k_1 g k_2) = f(g).$$

C is an algebra with respect to *convolution*:

$$(f_1 * f_2)(x) = \int_G f_1(xg) f_2(g^{-1})\, dg.$$

Let us record some basic facts about $C_c(K\backslash G/K)$.

(1) C is a commutative algebra. Indeed, any unitary representation $\rho : G \to \mathscr{U}(\mathcal{H})$ of G $\bar{\varphi} : C \to \mathrm{End}(\mathcal{H})$ gives rise to $\bar{\varphi} : C \to \mathrm{End}(\mathcal{H})$ such that

$$\bar{\varphi}(f)(v) = \int_G f(g)\rho(g)(v)\, dy.$$

(2) If $\mathcal{H}^K$ is the K-invariant subspace of $\mathcal{H}$, then

$$\bar{\rho}(C)(\mathcal{H}^k) \subseteq \mathcal{H}^k.$$

(3) If $(\mathcal{H}, \rho)$ is a unitary representation of G, then $\dim(\mathcal{H}^K) \leq 1$. ρ is called *spherical* if $\dim \mathcal{H}^K = 1$.
(4) If ρ is spherical, then ρ is completely determined by $\bar{\rho}$.

(1), (2), and (3) imply that C is a *Hecke algebra*. $\bar{\varphi}$ can be thought of as a map into $\mathbb{C}$; in this manner, $\bar{\varphi}$ can be thought of as a *character*.

Let $\bar{\delta} = 1_{K\left(\begin{smallmatrix} P & 0 \\ 0 & 1 \end{smallmatrix}\right)K}$, which is in C.

Claim. *C is generated as an algebra by $\bar{\delta}$.*

C acts on $L^2(G/K)$. Indeed, if $f_1 \in C$ and $f_2 \in L^2(G/K)$, then $f_2 * f_1 \in L^2(G/K)$. Now, if $\delta(f_2) = f_2 * \bar{\delta}$, then

$$\delta(f)(v) = \sum_{w \sim v} f(w).$$

Now, if Λ is a discrete cocompact discrete subgroup of G, then $L^2(\Lambda\backslash G)^K = L^2(\Lambda\backslash G/K)$. Therefore, there exists a one-to-one correspondence between spherical irreducible subrepresentations of $L^2(\Lambda\backslash G)$ and eigenvectors of the adjacency operator δ acting on $L^2(\Lambda\backslash G/K)$. In fact, there is a one-to-one correspondence between tempered representations and eigenvalues of δ on $\mathcal{H}^K$ that are less than or equal to $2\sqrt{p}$ in absolute value.

4.2. Bruhat–Tits Buildings. We begin by considering a simpler case. Let $U = \mathbb{F}_p^d$. Define $X(0)$ to be all proper subspaces of U. $\{W_0, \dots, W_t\}$ is a *t-cell* if it is a *flag*, i.e.,

$$W_0 \subseteq W_1 \subseteq \cdots \subseteq W_t.$$

Now, $\{W_0, \dots, W_{d-1}\}$ is a maximal flag, and so

$$\dim(\mathcal{S}(\mathbb{F}_p, d)) = d - 2.$$

We now consider $F = \mathbb{Q}_p/\mathbb{F}_p(t)$. We define a *Bruhat–Tits building* $\mathcal{B}_d(F) = \mathcal{B}$ as follows. Given a basis $\{\alpha_1, \dots, \alpha_d\}$ of $U = F^d$, we let $L = \mathbb{Z}_p\alpha_1 + \cdots + \mathbb{Z}_p\alpha_d$ be a lattice. Declare $L_1 \sim L_2$ in case there exists a nonzero $\mu \in F$ such that $\mu L_1 = L_2$. We define the vertex set of $\mathcal{B}$ to be the collection $\{[L] : L \text{ is a lattice}\}$.

We say that $[L_0], \dots, [L_t]$ is a t-dimensional cell if there exist representatives $L_i \in [L_i]$ such that $[L_0], \dots, [L_t]$ form a "flag":

$$pL_0' \subseteq L_t' \subseteq \cdots \subseteq L_2' \subseteq L_1' \subseteq L_0.$$

Observe that $L_0'/pL_0' \cong \mathbb{F}_p^d$. Similarly as above, we see that

$$\dim(\mathcal{B}) = d - 1.$$

Set $G = PGL_d(F)$. G acts transitively on bases. $Z = Z(G)$, the scalar matrices, preserve equivalence classes. $G = \tilde{G}/Z(G) = PGL_d(F)$ acts on the vertices of $\mathcal{B}$ as automorphisms of the building. If $L_0 = \mathbb{Z}_p e_1 + \cdots + \mathbb{Z}_p e_d$ is the standard lattice, then

$$K = PGL_d(\mathbb{Z}_p) = \mathrm{stab}_G([L_0]).$$

DEFINITION 4.3. The vertices of $\mathcal{B}$ come with a *color* in $\mathbb{Z}/d\mathbb{Z}$ defined as follows: given $L = g(L_0)$, $g \in GL_d(F)$, we set

$$\tau([L]) = \mathrm{val}(\det(g)) \bmod d \ \in \mathbb{Z}/d\mathbb{Z}.$$

The action of G does not preserve the colors. Also, if $[L_1] \xrightarrow{e} [L_2]$, then $\tau(e) = \tau(e^+) - \tau(e^-) \in \mathbb{Z}/d\mathbb{Z}$. Finally, every camber (maximum cell) contains a unique point of every color. We also remark that the color of edges is preserved by the automorphisms.

4.3. Outline of Friday's lecture. Let $G = PGL_d(\mathbb{Q}_p)$, $PGL_d(\mathbb{F}_p(t))$. Let $K = PGL_d(\mathbb{Z}_p)$. G/K is a *Bruhat–Tits building*, a $(d-1)$-dimensional simplicial complex

If Λ is a cocompact lattice in G, then $\Lambda\backslash G/K$ is a finite simplicial complex.

THEOREM 4.4 (Lafforgue). *If $\Lambda = \Gamma(I)$ is a congruent subgroup of $\Gamma \leq G = PGL_d(\mathbb{F}_p(t))$, then every infinite-dimensional, irreducible, spherical subrepresentation of $L^2(\Gamma(I)\backslash G)$ is tempered.*

CHAPTER 4

Roy Meshulam: Random Simplical Complexes

1. Topology of the Binomial Model (Day 2, Period 4)

1.1. Random graphs. Let $G(n,p)$ be the space of all random graphs on $[n] = \{1,\dots,n\}$ with independent edge probabilities p. Random graphs are ubiquitous. They serve as models for a multitude of natural phenomena, e.g., phase transition problems of statistical physics. They are also useful for showing existence results.

THEOREM 1.1 (Erdős, 1948). *A typical $G \in G(n, \frac{1}{2})$ will contain neither a clique nor an independent set on $k = 2\log_2 n$ vertices.*

Current best explicit constructions achieve only $k = 2^{(\log n)^\varepsilon}$ (for each fixed $\varepsilon > 0$).

THEOREM 1.2 (Erdős–Rényi, 1958). *For any function $\omega(n)$ that tends to infinity*[1]*,*

$$\lim_{n\to\infty} \mathbb{P}[G \in G(n,p) : G \text{ connected}] = \begin{cases} 0 & \text{if } p = \frac{\log n - \omega(n)}{n} \\ 1 & \text{if } p = \frac{\log n + \omega(n)}{n} \end{cases}$$

Moreover,

$$\lim_{n\to\infty} \mathbb{P}[G \in G(n,c/n) : G \text{ acyclic}] = \begin{cases} 0 & \text{if } c > 1 \\ \sqrt{1-c}\cdot e^{\frac{2c+c^2}{4}} & \text{if } c < 1. \end{cases}$$

REMARK. The first part of the theorem does not give a definite answer when $p = \frac{\log n}{n}$.

We shall study high-dimensional generalizations of the above theorem.

1.2. The k-dimensional Erdős-Rényi model. Let Y be a simplicial complex. $Y^{(i)}$ denotes the i-dim skeleton of Y. $Y(i)$ denotes the oriented i-dimensional simplices of Y. We set $f_i(Y) = |Y(i)|$. Δ_{n-1} is the $(n-1)$-dimensional simplex on $V = [n]$.

We define $Y_k(n,p)$ to be the probability space of all complexes

$$\Delta_{n-1}^{(k-1)} \subseteq Y \subseteq \Delta_{n-1}^{(k)}$$

with probability distribution

$$\mathbb{P}[Y] = p^{f_k(Y)}(1-p)^{\binom{n}{k+1} - f_k(Y)}.$$

Compare the following results with Theorem 1.2.

[1]extremely slowly, in practice

THEOREM 1.3 (Linial–Meshulam, 2003; Meshulam–Wallach, 2006). *Fix an integer $k \geq 1$ and a finite abelian group R. For any function $\omega(n)$ that tends to infinity,*

$$\lim_{n\to\infty} \mathbb{P}[Y \in Y_k(n,p) : \tilde{H}_{k-1}(Y;R) = 0] = \begin{cases} 0 & \text{if } p = \frac{k\log n - \omega(n)}{n} \\ 1 & \text{if } p = \frac{k\log n + \omega(n)}{n}. \end{cases}$$

THEOREM 1.4 (Hoffman–Kahle–Paquette, 2013). *If $p > \frac{80k\log n}{n}$, then asymptotically almost surely $\tilde{H}_{k-1}(Y;\mathbb{Z}) = 0$.*

Here are some details. Let us say that $\sigma \in \Delta_{n-1}(k-1)$ is *isolated* in $Y \in Y_k(n,p)$ if it is not contained in any $\tau \in Y(k)$. If σ is isolated in Y, then $1_\sigma \in \tilde{H}^{k-1}(Y)$, and $1_\sigma \neq 0$. Now, if $p = \frac{k\log n - \omega(n)}{n}$, then

$$\mathbb{E}[\text{number of isolated } \sigma\text{'s}] = \binom{n}{k}(1-p)^{n-k}$$

tends to ∞ as $n \to \infty$. A second moment argument then implies that

$$\mathbb{P}[\tilde{H}_{k-1}(Y;R) \neq 0] \to 1.$$

We can also look at different generalizations of connectivity.

THEOREM 1.5 (Babson–Hoffman–Kahle, 2010). *For any $\varepsilon > 0$,*

$$\lim_{n\to\infty} \mathbb{P}[Y \in Y_2(n,p) : \pi_1(Y) = \{1\}] = \begin{cases} 0 & \text{if } p = n^{-\frac{1}{2}-\varepsilon}; \\ 1 & \text{if } p = n^{-\frac{1}{2}+\varepsilon}. \end{cases}$$

THEOREM 1.6 (Meshulam, 2011). *If $\{G_n\}_{n=1}^\infty$ is a sequence of finite groups with $|G_n| \leq n^c$, then*

$$\lim_{n\to\infty} \mathbb{P}\left[Y \in Y_2\left(n, \frac{(3c+6)\log n}{n}\right) : \mathrm{Hom}(\pi_1(Y), G_n) = \{1\}\right] = 1$$

Here are some details for the Babson–Hoffman–Kahle bound. For $Y \in Y_2(n,p)$ and $u \in [n]$, we let $Y_u = \mathrm{St}_Y(u)$. Then $Y_u \cap Y_v = \mathrm{St}_Y(uv) \cup G$. It follows that, if $p \geq \sqrt{\frac{10\log n}{n}}$, then $Y_u \cap Y_v$ is connected for all u and v. It now follows from the nerve lemma that $\pi_1(Y) = \{1\}$.

Let us now consider **hyperbolic groups**. Let $F(X)$ be the free group on a set X. Fix $w \in F(X)$ such that $\bar{w} = 1$ is in $G = \langle X|R\rangle$, where R is a finite set of relations. We define the *area* $A(w)$ of w to be the minimal n such that

$$w = (u_1^{-1} r_1^{\varepsilon_1} u_1) \cdots (u_n^{-1} r_n^{\varepsilon_n} u_n)$$

for some $u_i \in F(X)$, $r_i \in R$, and $\varepsilon_i \in \{-1, 1\}$. We say that a group G is *hyperbolic* [fill in later]

Some examples of hyperbolic groups: finite groups, free groups, and fundamental groups of surfaces of genus at least 2, i.e.,

$$G = \langle a_1, b_1, \ldots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g]\rangle.$$

We also remark that $\mathbb{Z}^2$ is not hyperbolic.

We now let X be a 2-dimensional simplicial complex, γ a simplicial null-homotopic loop in X of length $|\gamma|$, and $A_X(\gamma)$ the minimal number of simplices in a filling of γ. The *isoperimetric constant* is the quantity

$$I(X) = \inf\left\{\frac{|\gamma|}{A_X(\gamma)} : \gamma \sim 1\right\}.$$

THEOREM 1.7. *$I(X) > 0$ if and only if $\pi_1(X)$ is hyperbolic.*

Here is a local-to-global principal that is useful for the proof:

THEOREM 1.8 (Gromov). *Let $c > 0$ and let X be a 2-dimensional complex that satisfies $I(S) \geq c$ for all pure $S \subseteq X$ such that $f_2(S) \leq 10^6 c^{-2}$. Then $I(X) \geq 10^{-2} c$.*

There is also a density invariant:

THEOREM 1.9 (Babson–Hoffman–Kahle). *Define*

$$\tilde{\mu}(X) = \min \left\{ \frac{f_0(Z)}{f_2(Z)} : Z \subseteq X \right\}.$$

If $\tilde{\mu}(X) > \frac{1}{2}$, then X is homotopic to a wedge of circles, 2-spheres, and projective planes. Moreover, for each $\varepsilon > 0$, there exists a $c_\varepsilon > 0$ such that $\tilde{\mu}(X) > \frac{1}{2} + \varepsilon$ implies $I(X) > c_\varepsilon$.

THEOREM 1.10 (Babson–Hoffman–Kahle). *$p = o(n^{-\frac{1}{2}-\varepsilon})$ implies*

$$\mathbb{P}[I(Y) > 0] = 1 - o(1).$$

SKETCH OF PROOF. Let S be a pure complex such that $f_2(S) \leq 10^6 c_\varepsilon^{-2}$. If $\tilde{\mu}(S) \leq \frac{1}{2} + \varepsilon$, then there exists a subcomplex $Z \subseteq S$ such that $f_0(Z) \leq (\frac{1}{2} + \varepsilon) f_2(Z)$ and hence

$$\mathbb{P}[Y \supseteq S] \leq n^{f_0(Z)} p^{f_2(Z)} = n^{(\frac{1}{2}+\varepsilon) f_2(Z)} \cdot o(n^{-\frac{1}{2}-\varepsilon})^{f_2(Z)} = o(1).$$

Therefore Y asymptotically almost surely satisfies the following condition: $S \subseteq Y$ and $f_2(S) \leq 10^6 c_\varepsilon^{-2}$ imply $\tilde{\mu}(S) > \frac{1}{2} + \varepsilon$, which then implies $I(S) > c_\varepsilon$. It now follows from the local-to-global principle that $I(Y) > 10^{-2} c_\varepsilon$. $\square$

CHAPTER 5

Izhar Oppenheim: High-dimensional Expanders from a 1-Dimensional Perspective

1. Day 4, Period 5

Let us first review the basic notions and results of the theory of expanders. See Chapter

Let $X = (V, E)$ be a graph. Define $\Delta_0 : l^2(V) \to l^2(V)$ by setting

$$\Delta_0 \phi(v) = \phi(v) = \frac{1}{\deg(v)} \sum_{u \sim w} \phi(u).$$

Then Δ_0 is positive-definite, with eigenvalues

$$0 = \mu_0(X) \leq \mu_1(X) \leq \cdots \leq \mu_{n-1}(X).$$

Recall that X has λ-*spectral expansion* if $\mu_1(X) \geq \lambda$. This implies that $\tilde{H}^0(X, \mathbb{R}) = 0$ by Cheeger's inequality.

Recall also that X has *two-sided* (λ, κ)-*expansion* if $\mu_1(X) \geq \lambda$ and $\mu_{n-1}(X) \leq \kappa$. This leads to the mixing lemma (Chapter 1, Theorem 1.2).

Finally, we recall that if X has λ-spectral expansion and if X is bipartite, then we obtain the bi-partite mixing lemma.

Definition 1.1. Let X be a simplicial complex of dimension d. For each $\tau \in X(k)$, we define X_τ to be the collection of all $\sigma \in X(i)$ such that $\sigma \cap \tau = \varnothing$ and that $\sigma \cup \tau \in X(k+i+1)$.

From now on, X is a pure d-dimensional complex with all the links of X dimension at least 1 are connected.

Definition 1.2. For $\lambda > \frac{d-1}{d}$, we say that X has λ-*local spectral expansion* if, for all $\tau \in X(d-2)$,

$$\mu_1(X_\tau) \geq \lambda.$$

This implies that, for all $0 \leq k \leq d-1$, we have $\tilde{H}^k(X, \mathbb{R}) = 0$, a Cheeger-type inequality.

Definition 1.3. For $\lambda > \frac{d-1}{d}$ and $\kappa < 2$, we say that X has λ-*two-sided-local spectral expansion* if, for all $\tau \in X(d-2)$,

$$\mu_1(X_\tau) \geq \lambda \quad \text{and} \quad \mu_{n-1}(X_\tau) \leq \kappa.$$

This gives us a mixing-type result. We also note that if X has λ-local spectral expansion and if X is $(d+1)$-partite, then we obtain the $(d+1)$-partite mixing lemma.

The above results are derived using the following methods:

(1) Weights

(2) Garland's method
(3) Spectral decent

CHAPTER 6

János Pach: Semialgebraic Combinatorics

1. Semi-algebraic Extremal Graph Theory (Day 2, Period 2)

Here are the cornerstones of extremal graph theory:

- Ramsey theorem (1930) and Erdös–Szekeres theorem (1935)
- Turán theorem (1941) and Kővári–Sós–Turá theorem (1954)
- Szemerédi regularity lemma (1978; Chapter 1, Theorem 1.6)

1.1. Ramsey theory. Ramsey theory is discussed in detail in Chapter 8, Section 1.1 for basic terminology.

Theorem 1.1 (Erdős–Szekeres, 1935).

$$\frac{1}{2}\log n \leq r_2(n) \leq 2\log n.$$

Theorem 1.2 (Erdős–Hajnal–Rado, 1965).

$$c\log\log n \leq r_3(n) \leq c'(\log n)^{1/2}$$

Here is an application:

Theorem 1.3. *Let $s = s(n)$ denote the largest number such that every set of n segments in $\mathbb{R}^2$ has s elements that are disjoint or all intersect. Then*

$$s(n) \geq r_2(n) \sim c\log n.$$

In this case, however, a non-Ramsey-theory proof gives a much better bound (LMPT 1994):

$$s(n) \geq n^{1/5}.$$

Here is another application:

Theorem 1.4. *We let $c(n)$ denote the largest number c such that every set of n points in $\mathbb{R}^2$ has c points that form a convex polygon. Then*

$$c(n) \geq r_3(n) \geq c\log\log n.$$

A different proof that makes use of a "hidden version" of the Ramsey theorem gives a better bound (Erdős–Szekeres, 1935):

$$c(n) \geq \frac{1}{2}\log n$$

1.2. Turán-type problems. Turán-type problems are discussed in detail in Chapter 8, Section 1.2 and Section 2.1

THEOREM 1.5 (Kővár–Sós–Turán, 1954). *Let $G = (V(G), E(G))$ be a graph with n vertices such that $G \not\supseteq K_{r,r}$ (the complete bipartite graph with r vertices in each class). Then*

$$|E(G)| \leq c_r^{2-1/r}.$$

COROLLARY 1.6. *Let $I(n)$ denote the maximum number of incidences between n points and n lines in $\mathbb{R}^2$. Then*

$$I(n) \leq c_2 n^{2-1/2} = c_2 n^{3/2}$$

A newer proof that makes use of the Kővár–Sós–Turán theorem in a roundabout way gives the optimal bound (Szemerédi–Trotter, 1983):

$$I(n) \sim cn^{4/3} \tag{1.7}$$

1.3. Semi-algebraic graphs and hypergraphs.

DEFINITION 1.8. A subset $A \subseteq \mathbb{R}^d$ is *semi-algebraic* if there exist polynomials $f_1, \dots, f_t : \mathbb{R}^d \to \mathbb{R}$ of degree at most t and a Boolean formula ϕ such that

$$A = \{x \in \mathbb{R}^d : \phi[f_1(x) \geq 0, \dots, f_t(x) \geq 0]\}.$$

The *description complexity* of A is the quantity $\max(d, t)$.

DEFINITION 1.9. A *semi-algebraic graph* is a graph $G = (V, E)$ with a semi-algebraic set $A \subseteq \mathbb{R}^d \times \mathbb{R}^d$ such that $xy \in E$ if and only if $(x, y) \in A$.

See Chapter 8, Definition 1.1 for the definition of a k-graph.

DEFINITION 1.10. A *semi-algebraic k-graph* is a k-graph $G = (V, E)$ with a semi-algebraic set $A \subseteq (\mathbb{R}^d)^k$ such that $(x_1, \dots, x_k) \in E$ if and only if $(x_1, \dots, x_k) \in A$.

EXAMPLE 1.11. Intersection graphs of segments, cubes, balls, and so on are semi-algebraic graphs.

EXAMPLE 1.12. Clockwise orientations of point-triples in $\mathbb{R}^2$ are semi-algebraic 3-graphs.

EXAMPLE 1.13. A collection of triples of balls in $\mathbb{R}^3$ admitting a line transversal is a semi-algebraic graph.

We now introduce Ramsey-theoretic notions for this framework.

DEFINITION 1.14. $R_k(n)$ is the minimum R such that in every 2-coloring of all k-tuples of an R-element set, there is a monochromatic subset of size n.

DEFINITION 1.15. $R_k^t(n)$ is the minimum R such that in every 2-semi-algebraic-colorings with description complexity bounded above by t of all k-tuples of an R-element set, there is a monochromatic subset of size n.

Here is a result by Erdős–Rado, 1952:

$$R_k^t(n) \leq R_k(n) \leq T(k)^{cn}$$

THEOREM 1.16 (Conlon–Fox–Pach–Sudakov–Suk, 2013). *(statement?)*

THEOREM 1.17 (Alon–Pach–Pinchasi–Radoičić–Sharir, 2005). *Let $G = (V, E)$ be a semi-algebraic graph of description complexity t with $|V| = n$. There exist $V_1, V_2 \subseteq V$ with $|V_1|, |V_2| \geq \varepsilon n$ such that $V_1 \times V_2 \subseteq E$ or $(V_1 \times V_2) \cap E = \varnothing$. (Here $\varepsilon = \varepsilon(t) > 0$ depends only on t.*

COROLLARY 1.18 (Pach–Solymosi, 2001). $R^t(n) \leq n^{c(t)}$

We now turn to semi-algebraic Turán theory.

THEOREM 1.19 (Mantel, 1907). *Let $\text{ex}(n, K_3)$ denote the maximum number of edges that a K_3-free graph of n vertices can have. Then $\text{ex}(n, K_3) = \lfloor n^2/4 \rfloor$.*

Let $\text{ex}^t(n, K_3)$ denote the maximum number of edges that a K_3-free semi-algebraic graphs of description complexity bounded above by t of n vertices can have.

THEOREM 1.20 (Kővári–Sós–Turán, 1954). *Let $G = (V_1 \cup V_2, E)$ be a $K_{r,r}$-free bipartite graph with $|V_1| = |V_2| = n$. Then*

$$|E(G)| \leq c_r n^{2-1/r}.$$

THEOREM 1.21 (Fox–Pach–Sheffer–Suck–Zahl, 2014). *Let $G = (V_1 \cup V_2, E)$ be a $K_{r,r}$-free semi-algebraic bipartite graph with $V_1, V_2 \subseteq \mathbb{R}^d$, $|V_1| = |V_2| = n$. Then*

$$|E(G)| \leq c_{r,t} n^{4/3}$$

if $d = 2$, and

$$|E(G)| \leq c_{r,t,\varepsilon} n^{2-2/(d+1)+\varepsilon}$$

if $d \geq 2$ and $\varepsilon > 0$. Here t is the description complexity of G.

This result implies the Szemerédi–Trotter theorem (1.7).

1.4. Regularity theory. Regularity lemmas are discussed in detail in Chapter 1, Section 1.

THEOREM 1.22 (Szemerédi regularity lemma, 1978). *For any $\varepsilon > 0$, there exists an integer K_ε such that the vertex set of every sufficiently large graph G has a partition into K almost equal parts $V_1 \cup \cdots \cup V_k$, $K \leq K_\varepsilon$ with the property that all but at most εK^2 pairs (V_i, V_j) are ε-regular if and only if they behave like random graphs with an ε error.*

THEOREM 1.23 (Semi-algebraic regularity lemma). *For any $\varepsilon > 0$ and t, there exists an integer $K_{\varepsilon,t}$ such that the vertex set of every sufficiently large semi-algebraic graph of description complexity at most t has a partition into K almost equal parts $V_1 \cup \cdots \cup V_k$, $K \leq K_{\varepsilon,t}$ with the property that all but at most εK^2 pairs (V_i, V_j) induce an empty or a complete bipartite graph if and only if $V_i \times V_j \subseteq E(G)$ or $(V_i \cap V_j) \cap E(G) = \varnothing$.*

Now, Theorem 1.17, combined with the semi-algebraic regularity lemma, yields the following *selection theorem*:

THEOREM 1.24 (Alon–Pach–Pinchasi–Radoičić–Sharir, 2005). *For any sets of vectors $U, V \subseteq \mathbb{R}^d$, one can select $U' \subseteq U$, $V' \subseteq V$ with $|U'| \geq \frac{1}{2^{d+1}}|U|$, $|V'| \geq \frac{1}{2^{d+1}}|V|$ such that*

- *$\langle u, v \rangle \geq 0$ for all $u \in U'$ and $v \in V'$, or*
- *$\langle u, v \rangle < 0$ for all $u \in U'$ and $v \in V'$.*

We refer the reader to Chapter 1, Lemma 2.12 for the semi-algebraic hypergraph regularity lemma.

CHAPTER 7

Ori Parzanchevski: High-Dimensional Laplacians and Expansion

1. Cohomology and Spectrum (Day 1, Period 2)

1.1. Dimension 1. Let $X = (V, E)$ be a graph. For our purposes, E consists of nonoriented edges. In contrast, we write $E_\pm$ to denote the collection of *oriented edges*, i.e., the set of ordered pairs (v, w) where $\{v, w\} \in E$. It follows at once that $|E_\pm| = |E|$.

The space of *1-forms* (also called *flows*, *vector fields*, or *cochains*) on X is the set of all functions $f : E_\pm \to \mathbb{R}$ such that $f(wv) = -f(vw)$ for all $wv = (w, v) \in E_\pm$. The *dimension* of $\Omega^1(X)$ is defined to be the cardinality $|E|$ of the edge set E.

For notational purposes, we define Ω^0 to be the set of all functions $f : V \to \mathbb{R}$ and Ω^{-1} to be the set of all functions $g : \{\varnothing\} \to \mathbb{R}$. We shall identify Ω^{-1} with $\mathbb{R}$.

The *coboundary map* $d_0 : \Omega^0(X) \to \Omega^1(X)$ takes $f : \Omega^0(X) \to \mathbb{R}$ to $\tilde{f} : \Omega^1(X)$ given by the formula $\tilde{f}(ab) = f(a) - f(b)$. The *boundary map* $\partial_1 : \Omega^1(X) \to \Omega^0(X)$ takes $f : \Omega^1(X) \to \mathbb{R}$ to $\bar{f} : \Omega^0(X) \to \mathbb{R}$ given by the formula

We define an inner product on $\Omega^1(X)$ as follows:

$$\langle f, g \rangle = \sum_{e \in E_\pm} f(e)g(e).$$

With this inner product, we see that $d_0 = \partial_1^*$, in the sense that

$$\langle d_0 f, g \rangle = \langle f, \partial_1 g \rangle$$

for all $f \in \Omega^0$.

Similarly, we define the *coboundary map* $d_{-1} : \Omega^{-1}(X) \to \Omega^0(X)$ to be the map that sends each constant $c \in \mathbb{R}$ to the constant map $c\mathbf{1}$. Considering d_{-1} as a matrix operator, we can define the *boundary map* $\partial_0 : \Omega^0(X) \to \Omega^{-1}(X)$ to be the transpose of d_{-1}.

Observe that $\operatorname{im} d_{-1}$ is the set of all constant maps on V. Since $\ker d$ is the set of all *locally constant* maps on V, viz., constant on every connected component of X, it follows at once that $\operatorname{im} d_{-1} \subseteq \ker d_0$, with the equality achieved if and only if X is connected. The *zeroth cohomology module of* X is defined to be the quotient space

$$H^0(X) = \ker d_0 / \operatorname{im} d^{-1}.$$

Observe that the dimension of H^0 is $\dim \ker d_0 - \dim \operatorname{im} d^{-1}$.

We define the *Laplacian* $\Delta_0 = \partial_1 d_0 : \Omega^0(X) \to \Omega^0(X)$. Observe that

$$(\Delta_0 f)(v) = \deg(v)f(v) - \sum_{w \sim v} f(w).$$

In particular, if X is k-regular, then $\Delta_0 = kI - A$. Observe that

$$\ker \Delta_0 = \ker \partial_1 d_0 = \ker d_0^* d_0 = \ker d_0.$$

To see this, we note that, if $f \in \ker d_0^* d_0$, then

$$0 = \langle d_0^* d_0 f, f \rangle = \langle d_0 f, d_0 f \rangle,$$

and so $f \in \ker d_0$.

In conclusion, we have the following inclusion relation:

$$\{c\mathbf{1} : c \in \mathbb{R}\} = \operatorname{im} d_{-1} \subseteq \ker d_0 = \ker \Delta_0. \tag{1.1}$$

Remark (Homology!). We pause here to note that

$$\operatorname{im} \Delta_0 = \operatorname{im} \partial_1 \partial_1^* = \operatorname{im} \partial_1,$$

and $\operatorname{im} \partial_1$ is the set of all maps $f : V \to \mathbb{R}$ that sum to zero on every connected component. □

What is in $\operatorname{spec} \Delta_0$? (1.1) implies that $0 \in \operatorname{spec} \Delta_0$; this is the *trivial part* of $\operatorname{spec} \Delta_0$. We call $\operatorname{spec} \Delta_0 \setminus \{0\}$ the *nontrivial part* of $\operatorname{spec} \Delta_0$. Since $(\operatorname{im} d^{-1})^\perp = \ker \partial_0$, we see that the nontrivial part of $\operatorname{spec} \Delta_0$ consists of the eigenvalues that come from functions that sum to zero. All in all, we have the following in $\operatorname{spec} \Delta_0$:

- $\operatorname{im} d_{-1}$: 0
- $\ker \partial_0$: the zero-sum maps.
  - $\ker d_0$: 0,...,0.
  - $\operatorname{im} \partial_1$: nonzero.

From this, we obtain the following *Hodge decomposition* of $\Omega^0(X)$:

$$\Omega^0 = \operatorname{im} d_{-1} \oplus (\ker \partial_0 \cap \ker d_0) \oplus \operatorname{im} \partial_1.$$

We define $\lambda_0(X)$ to be the minimum of all elements of $\operatorname{spec} \Delta_0$ restricted to the "zero-sum maps" part. Note that $\lambda_0(X) = 0$ if X is disconnected and $\lambda_0(X) > 0$ if X is connected.

1.2. Dimension 2. Let $X = (V, E, T)$ be a *two-dimensional simplicial complex*, which is a graph (V, E) with a set T of *triangles*, which are sets $\{v, u, w\}$ such that $\{v, w\}, \{v, u\}, \{v, w\} \in E$. Similarly as in the 1-dimensional case, we define the collection $T_\pm$ of *oriented triangles* to be the ordered variant of T. We note that $|T_\pm| = |T|$.

The space of *2-forms* on X is defined to be the set $\Omega^2(X)$ of functions $f : T_\pm \to \mathbb{R}$ such that

$$f(vuw) = f(uvw)$$

for all $vuw \in T_\pm$.

[fill in later: definition of d_1 and ∂_2]

Observe that

$$\ker d_1 = \left\{ f \in \Omega^1(X) : \int_\Delta f = 0 \text{ over all triangles } \Delta \right\}.$$

On the other hand,

$$\operatorname{im} d_0 = \left\{ f \in \Omega^1(X) : \oint f = 0 \text{ over all closed paths} \right\}.$$

The *first cohomology module of* X is defined to be the quotient space

$$H^1(X) = \ker d_1 / \operatorname{im} d_0.$$

$\ker d_1$ is commonly known as the space of *closed forms*; $\operatorname{im} d_0$ is commonly known as the space of *exact forms*. In other words, if $H^1(X) \neq 0$, then there exists a function $f \in\in \Omega^1(X)$ such that $\int_\Delta f = 0$ over all triangles Δ, but $\oint f \neq 0$ for some closed path. This means there is a closed path that is not *triangulizable*; in other words, there is a closed path on X that cannot be realized as the boundary of a triangle. Intuitively, this implies that X "has holes".

We now define the *Laplacian* to be the map $\Delta_1 = \partial_2 d_1$. The trivial part of $\operatorname{spec} \Delta_1$ comes from $\operatorname{im} d_0$, and the nontrivial part of $\operatorname{spec} \Delta_1$ comes from $(\operatorname{im} d_0)^\perp = \ker \partial_1$, the Kirchoff flows. Since

$$\operatorname{im} d_0 \subsetneq \ker \Delta_1 = \ker d_1$$

if and only if $H^1(X) \neq 0$, we see that there exists a Kirchoff closed form if and only if there are holes on the complex X. This is due to Beno Eckmann.

The Hodge decomposition in this case is as follows:

$$\Omega^1 = \operatorname{im} d_0 \oplus (\ker d_1 \cap \ker \partial_1) \oplus \operatorname{im} \partial_2.$$

1.3. General Case. In general, we consider j-cells $X^j_\pm$ with orientation. We can then define $\Omega^j(X)$ and develop an analogous theory with respect to the map

$$(df)(\sigma) = \sum_{v : v \cup \sigma \in X^{j+1}} f(v \cup \sigma).$$

2. Spectrum, Cheeger's Inequalities, and Mixing (Day 4, Period 4)

Let $X = (V, E, T)$ be a simplicial complex and take $\lambda_1 = \min \operatorname{spec} \Delta_1 |_{Z_1}$. Observe that $\lambda_1 = 0$ if and only if (V, E, T) has holes, i.e., $H^1 \neq 0$. What can we say about $\lambda_1 \gg 0$? Here is a multi-dimensional generalization of the *Cheeger constant* (Chapter 2, Definition 1.1):

$$h_1(X) = \min_{\substack{A \coprod B \coprod C = V \\ A, B, C = \varnothing}} \frac{|T(A, B, C)| n}{|A||B||C|}.$$

Recall that, if we take the one-dimensional Cheeger constant to be

$$h_0(X) = \min_{A \coprod B = V} \frac{|E(A, B)| n}{|A||B|},$$

then we have the following Cheeger inequality:

$$\frac{h_0^2}{8k} \leq \lambda_0 \leq h_0,$$

where k is the maximum degree of a vertex of X.

THEOREM 2.1 (Parzanchevski–Rosenthal–Tessler). *If $E = \binom{V}{2}$, then $\lambda_1 \leq h_1$. In dimension d, for $X^{d-1} = \binom{V}{d}$, we have that $\lambda_{d-1} \leq h_{d-1}$.*

PROOF. Given $A \coprod B \coprod C = V$, consider $f \in \Omega^1$. Given $a = |A|$, $b = |B|$, and $c = |C|$, we set

$$f(vw) = \begin{cases} a & v \in B \text{ and } w \in C \\ \vdots & \end{cases}$$

$f \in Z_1$, and so

$$(\partial_1 f)(v) = cb + b(-c) = 0.$$

Therefore,

$$\lambda_1 \leq \frac{\langle \Delta_1 f, f\rangle}{\langle f, f\rangle},$$

as $\lambda_1 \leq \operatorname{spec} \Delta_1$. It thus suffices to estimate the right-hand side.

Observe first that

$$\begin{aligned}\langle f, f\rangle &= \sum_e f(e)^2 \\ &= \sum_{c:A\to B} c^2 + \sum_{a:B\to C} a^2 + \sum_{b:C\to A} b^2 \\ &= abc^2 + ab^2c + a^2bc = (a+b+c)abc = nabc.\end{aligned}$$

We also see that

$$\begin{aligned}\langle \Delta_1 f, f\rangle &= \langle \partial_2 d_1 f, f\rangle \\ &= \langle d_1 f, d_1 f\rangle \\ &= \sum_{t\in T} (d_1 f)(t)^2 \\ &= |T(A,B,C)|n^2.\end{aligned}$$

It thus follows that

$$\lambda_1 \leq \frac{|T(A,B,C)|n^2}{nabc} \leq h_1.$$

□

Remark. There is a generalization for non-complete skeletions by Szedlák–Gundert and Parzanchevski–Golubev.

We also recall the expander mixing lemma (Chapter 1, Theorem 1.2): $\operatorname{spec} \Delta_0|_{Z_0} \subseteq [k-\varepsilon, k+\varepsilon]$, and so, for all A and B,

$$\left| |E(A,B)| - \frac{kab}{n} \right| \leq \sqrt{\varepsilon^2 ab}.$$

Here is a higher-dimensional generalization of the mixing lemma:

Theorem 2.2. *If* $\operatorname{spec} \Delta_1|_{Z_1} \subseteq [k-\varepsilon, k+\varepsilon]$ *for any* $k > 0$ *and* $\varepsilon > 0$*, and if* $E = \binom{V}{2}$*, then*

$$\left| |T(A,B,C)| - \frac{kabc}{n} \right| \leq (\varepsilon^3 abc)^{3/2}.$$

Proof. Let

$$1_{AB}(e) = \begin{cases} 1 & \text{if } e : A \to B \\ -1 & \text{if } e : B \to A \\ 0 & \text{otherwise.}\end{cases}$$

Then

$$-\langle \Delta 1_{AB}, 1_{AC}\rangle = -\langle \partial_2 d_1 1_{AB}, 1_{AC}\rangle = -\langle d_1 1_{AB}, d_1 1_{AC}\rangle$$

Now,

$$(d_1 1_{AB})(t) = \begin{cases} 1 & \text{if } t \in T(A,B,C) \text{ or } t \in T(A,B,V \smallsetminus (A,B,C)) \\ 0 & \text{otherwise.}\end{cases}$$

and

$$(d_1 1_{AC})(t) = \begin{cases} 1 & \text{if } t \in T(A, C, V \setminus (A \cup C)) \\ 0 & \text{otherwise.} \end{cases}$$

Observe that

$$\begin{aligned} |T(A,B,C)| &= \langle (kI - \Delta_1) 1_{AB}, 1_{AC} \rangle \\ &= \langle (kI - \Delta_1)(1_{AB}^{B^1} + 1_{AB}^{Z_1}), 1_{AC} \rangle. \end{aligned}$$

We estimate the two terms separately. Since $\mathrm{spec}(kI - \Delta_1)|_{Z_1} \subseteq [-\varepsilon, \varepsilon]$, Cauchy–Schwarz implies that

$$\begin{aligned} |\langle (kI - \Delta_1)(1_{AB}^{Z_1}), 1_{AC} \rangle| &\leq \|(kI - \Delta_1) 1_{AB}^{Z_1}\| \|1_{AC}\| \\ &\leq \varepsilon \|1_{AB}\| \|1_{AC}\| \\ &= \varepsilon \sqrt{ab}\sqrt{ac} = \varepsilon a \sqrt{bc}. \end{aligned}$$

As for the other term, we observe that

$$|\langle (kI - \Delta_1)(1_{AB}^{B^1}), 1_{AC} \rangle| = k \langle 1_{AB}^{B'}, 1_{AC} \rangle.$$

We claim that

$$\mathrm{Proj}_{B'} = \frac{d_0 \partial_1}{n},$$

which we call the *lower Laplacian* Δ^-. To see this, we note that

$$\Delta_1^- Z_1 = d_0 \partial_1 \ker \partial_1 = 0.$$

Observe that

$$\begin{aligned} \mathrm{spec}\, \Delta_1^- |_{B^1 = Z_1^\perp = \ker \partial_1^\perp = (\ker \Delta_1)^\perp} &= \mathrm{spec}\, \Delta_1^- \setminus \{0\} \\ &= \mathrm{spec}\, d_0 \partial_1 \setminus \{0\} \\ &= \mathrm{spec}\, \partial_1 d_0 \setminus \{0\} \\ &= \mathrm{spec}\, \Delta_0 \setminus \{0\} \\ &= \{n\} \end{aligned}$$

The last equality follows from the completeness of the graph.

Now,

$$\frac{k}{n} \langle \Delta_1^- 1_{AB}, 1_{AC} \rangle = \frac{k}{n} \langle \partial_1 1_{AB}, \partial_1 1_{AC} \rangle = \frac{kabc}{n}.$$

Finally,

$$\left| |T(A,B,C)| - \frac{kabc}{n} \right| \leq \varepsilon a \sqrt{bc} \cdot \varepsilon b \sqrt{ac} \cdot \varepsilon c \sqrt{ab}.$$

□

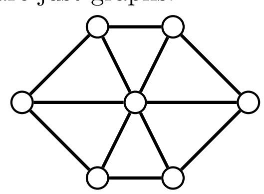
CHAPTER 8

Benny Sudakov: Extermal Results and Problems on Hypergraphs

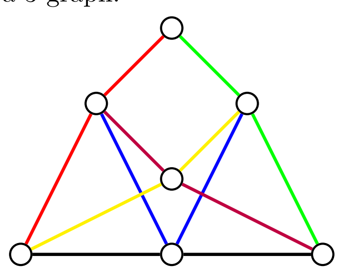
1. Part 1 (Day 1, Period 3)

Definition 1.1. A *k-graph* is a hypergraph $X = (V, E)$ such that each $e \in E$ is a subset of V of cardinality k.

Example 1.2. 2-graphs are just graphs:



Here is an example of a 3-graph:



1.1. Ramsey theory.

k-graphs arise, for example, in Ramsey theory. Let $r_k(s,t)$ be the smallest positive integer N such that every red-blue coloring of k-tuples on $[N]$ contains either a red s-clique or a blue t-clique.

If $k = 2$, then we have the following result:

Theorem 1.3 (Erdős–Szekeres). $2^{n/2} \le r(n,n) \le 2^{2n}$.

Sketch of proof. We first establish

$$r(s,t) \le r(s-1,t) + r(s,t-1)$$

and use this bound inductively to show that $r(s,t) \le 2^{s+t}$.

As for the lower bound, we let $N = 2^{n/2}$ and color edges on $[N]$ randomly with probability 1/2 for red and 1/2 for blue. Then the probability of the existence of monochromatic set of size n is bounded above by

$$\binom{N}{n} \cdot 2 \cdot 2^{-\binom{n}{2}} \ll 1.$$

□

This bound is 70 years old, and there hasn't been substantial improvement since then. The current constructions are random.

CONJECTURE 1.4 (explicit construction?). *Take* $[p]$ *to be the vertex set, where* p *is a prime number that is 1 mod 4. We declare* $x \sim y$ *if and only if* $x - y = z^2$. *This is the* Paley graph.

THEOREM 1.5 (Ajtai–Komlós–Szemerédi). $r(3,t) = \Theta\left(\frac{t^2}{\log t}\right)$.

REMARK. We recall that $f = \Theta(g)$ if and only if $f \le cg$ and $g \le c'f$ for some constants c and c'.

For $k = 3$, we can make use of random coloring to establish the lower bound

$$r_3(n,n) \ge 2^{cn^2}$$

for some constant c.

As for the upper bound, we have the following bound:

$$r_3(s,t) \le r_2(r_3(s-1,t), r_3(s,t-1)).$$

Of course, this isn't a terribly nice bound: we end up with an iterated exponential bound.

THEOREM 1.6 (Erdős–Rado, 1950s). $r_3(s,t) \le 2^{\binom{r(s,t)}{2}}$.
In particular,

$$r_3(n,n) \le 2^{2^{cn}}.$$

CONJECTURE 1.7. $r_3(n,n)$ *is of order* $2^{2^{cn}}$. *Indeed, if we use 4-coloring, then there is the Erdős–Hajnal bound*

$$r_3(n,n,n,n) \ge 2^{2^{cn}},$$

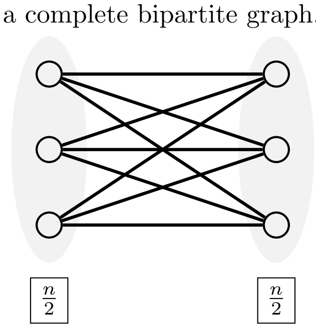
and this construction is not random.

We could try and look at a simpler case:

THEOREM 1.8 (Conlon–Fox–Sudakov). $2^{cn \log n} \le r_3(4,n) \le 2^{cn^2 \log n}$.

1.2. Turán-type problems. We define $\mathrm{ex}(n,H)$ to be the maximum number of edges in k-graph G on n vertices which contains no H.

EXAMPLE 1.9. Here is a complete bipartite graph.



A 1907 result of Mantel shows that

$$\mathrm{ex}(n,\Delta) = \frac{n^2}{4}.$$

Definition 1.10. We define the *chromatic number* $\chi(H)$ of a hypergraph H to be the minimal t such that the vertices of H can be t-colored without monochromatic edges.

Theorem 1.11 (Erdős–Stone). *Let H be a graph ($k = 2$). If $\chi(H) = r + 1$, then*

$$\mathrm{ex}(n, H) = \left(1 - \frac{1}{r}\right)\frac{n^2}{2} + o(n^2).$$

Let us now move on to the $k = 3$ case. Consider, for example, K_4^3, the 3-graph on 4 vertices that has all 4 triples. Even in a simple case like this, ex is unknown. In fact, let us define

$$\pi(H) = \lim_{n\to\infty} \frac{\mathrm{ex}(n, H)}{\binom{n}{3}},$$

so that determination of $\pi(H)$ is easier than that of $\mathrm{ex}(H)$. The following is open, with a bounty promised by Erdős.

Conjecture 1.12 (Turán). $\pi(K_4^3) = 5/9$.

We conjecture that there are exponentially many "extremal" configurations.

If k is an arbitrary positive integer, then we can consider K_{k+1}^k, the k-graph on $k + 1$ vertices that has all $k + 1$ k-tuples. The following bounds are known:

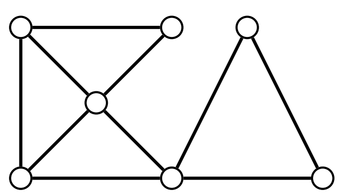
$$\frac{1}{k} \le 1 - \pi(K_{k+1})^k \le \frac{\log k}{2k}.$$

2. Part 2 (Day 1, Period 5)

2.1. Turán-type problems, continued.

Definition 2.1. Let $f(n, p, q)$ denote the maximum number of edges in 3-graph on n vertices such that every p vertices span q or more edges.

The most basic question we can ask is the (6,3)-problem. For this, every 6 vertices should have 3 or more edges. This, in particular, means there is nothing like this:



Therefore, we have the following inequality:

$$\binom{n}{2} \ge 3e(H).$$

We also do not have anything like this:

(triangles picture)

Indeed, we have the following result, which follows from the triangle removal lemma (Chapter 1, Theorem 1.9):

Theorem 2.2 (Ruzsa–Szemerédi). $f(n, 6, 3) = o(n^2)$.

The above result of Ruzsa–Szemerédi implies the following classical number-theoretic result of Roth:

THEOREM 2.3 (Roth). *Fix δ. Let $A \subseteq [N]$ such that $|A| = \delta N$. As $N \to \infty$, A contains a 3-term arithmetic progression.*

SKETCH OF PROOF. Take $A \subseteq [N]$ such that A contains no 3-term arithmetic progression. For $x \in [N]$ and $a \in A$, we consider $(x, x+a, x+2a) \subseteq [N]\times[2N]\times[3N]$, which gives rise to a hypergraph. Since A has no 3-term arithmetic progression, this hypergraph does not have anything of the following form:

(triangles picture)

Now, $e(H) = |A|N = o(n^2)$ by the Ruzsa–Szemerédi theorem, whence $|A| = o(N)$. □

Since $f(n, 6, 4)$ already yields interesting applications, we expect the proof of the following result to come with intriguing new methods.

CONJECTURE 2.4. $f(n, 7, 4) = o(n^2)$.

So far, we have examined $f(n, p, q)$ with $p - q = 3$. Let us now take a look at the $p - q = 2$ case. In this case, we can have a hypergraph with cn^2 edges, where c is an absolute constant.

CONJECTURE 2.5 (Erdős). *There exists a 3-graph on n vertices with cn^2 edges such that no set of size $p \le p_0$ spans at most $p_0 - 2$ edges. In fact, there should be a Steiner triple system with this property; an STS is a collection of $\frac{1}{3}\binom{n}{2}$ triples covering every pair exactly once.*

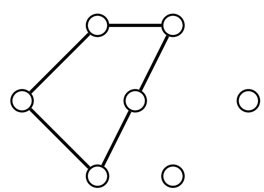
2.2. Cycles in hypergraph.

DEFINITION 2.6. A *cycle* in a graph is a collection

$$\{(v_1, v_2), \ldots, (v_{k-1}, v_k), (v_k, v_1)\}$$

of edges.

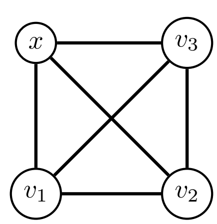
For example,



is a cycle.

DEFINITION 2.7. A *Berge cycle* in a hypergraph H is [fill in later]

For example,

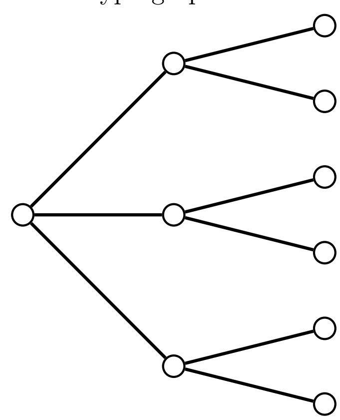


$$l_1 = (x, v_1, v_2), \qquad l_2 = (x, v_2, x_3), \qquad l_3 = (x, v_3, v_1)$$

is a Berge cycle.

DEFINITION 2.8. The *girth* $g(H)$ of a hypergraph H is the length of the shortest Berge cycle in H. The *chromatic number* $\chi(H)$ of H is the minimum t such that there exists a t-coloring of vertices with no monochromatic edges.

If the girth is large, then the hypergraph looks locally like a graph.



Now, trees are easy to color, so we are led to the following question: is $\chi(H)$ small if $g(H)$? Not so, as the next theorem shows:

THEOREM 2.9. *There exists a 3-graph H with large $g(H)$ and large $\chi(H)$.*

Let us sketch the proof. To this end, we let n denote the number of vertices. Put every triple in with probability $p = n^{-2+\frac{1}{n+1}}$. Then the number of short cycles is

$$\sum_{2\le t\le e} \binom{n}{t}\binom{n}{t} p^t \ll n^{2l}p^l = n^{l/e+1} < n.$$

Now, the number x of independent sets in the hypergraph is bounded above by

$$n^{1-\frac{1}{2(l+1)}} \operatorname{polylog} n \binom{n}{x} (1-p)^{\binom{x}{3}} \ll 1.$$

Therefore,

$$\chi(H) \ge \frac{n}{x} \approx n^{\frac{1}{2(l+1)}} = s.$$

So far, we have $g(H) \ge l$ and $\chi(H) \ge s$, with $n \approx s^{2(l+1)}$. If H is a graph, then $g(H) \ge l$ and $\chi(H) \ge s$ and at most s^{l+1} vertices, and we have an explicit construction.

Let us call G a (n, d, λ)*-graph* if G has n vertices, d-regular, and $|\lambda_i| \le \lambda$ for all $i \ge 2$. (See Chapter 3, Section 1 for notations and terminology.) We would like λ to be as small as possible: $\lambda \approx \sqrt{d}$. The *Hofman bound* is as follows: if $\alpha(H) \approx \frac{\lambda}{d} n$, then $\chi(H) \ge \frac{d}{\lambda}$.

We conclude this section with a problem from coding theory. A *mod2-cycle* in H is a collection of edges covering every vertex even number of times. The *size* of a cycle is the number of vertices covered at least once.

Feige suggested the following. Consider a 4-graph which has quadratic number of edges, say, $\binom{n}{2}$. The vertices are $\binom{n}{2}$ pairs of vertices of H. If (x, y) and (x', y') are two disjoint pairs of edges in H, then we declare $(x, y) \sim (x', y')$.

Let $N = \binom{n}{2}$. There are $\frac{1}{2}N$ ways to partition the vertices into disjoint pairs. Therefore, $3e(H) = 3N$, and so $G \supseteq G'$ is a graph with degree at least 3. It follows that in $O(\log n)$ steps we have a collision, a cycle.

CONJECTURE 2.10. *That even εn^2 4-tuple, give a short mod2-cycle (short* polylog n*?)*

CHAPTER 9

Uli Wagner: Coboundry Expansion and Topological Overlapping

1. Definitions (Day 2, Period 3)

Let $X = (V, E)$ be a graph. Denote by δS the set $E(S, V \smallsetminus S)$ of edges between S and $V \smallsetminus S$.

DEFINITION 1.1. X is said to be *η-edge expanding* ($\eta > 0$) if, for all $S \subseteq V$,

$$\frac{|\delta S|}{|S|} \geq \eta \frac{\min\{|S|, |V \smallsetminus S|\}}{|V|}. \tag{1.2}$$

The *edge-expansion constant* of X is the minimal η such that (1.2) holds for all $S \subseteq V$.

We consider X as a one-dimensional simplicial complex with underlying space $\|X\|$. X is η-edge expanding, and so, for all $f : \|X\| \to \mathbb{R}^1$, there exists a $p \in \mathbb{R}^1$ such that

$$|\{e \in E : f(e) \ni p\}| > \lambda |E|, \tag{1.3}$$

where $\lambda = \frac{\eta}{2}$. Note that edge expansion is not required for (1.3): a weaker hypothesis, such as having large bisection width, would suffice.

Generalizing, we now consider a d-dimensional simplicial complex X with underlying space $\|X\|$. We let X_k denote the set of all k-simplices of a finite simplicial complex X, so that $\|X\| = \bigcup_{\sigma \in X} \sigma$. We consider the space of *k-chains* $C_k(X) = C_k(X; \mathbb{F}_2)$ and the space of *k-cochains* $C^k(X) = C^k(X; \mathbb{F}_2)$, both of which are linearly isomorphic to $\mathbb{F}_2^{X_k}$. We also consider the map $\partial : C_k(X) \to C_{k-1}(X)$ given by the formula

$$\partial \sigma = \sum_{\substack{\tau \in X_{k-1} \\ \tau \subseteq \sigma}} \tau$$

for all $\sigma \in X_k$, and the map $\delta : C^k(X) \to C^{k+1}(X)$ given by the formula

$$(\delta c)(\sigma) = c(\partial \sigma).$$

for all $\sigma \in X_{k+1}$. We obtain the following cochains with coefficients in $\mathbb{F}_2$; here we set $X_{-1} = \{\varnothing\}$ for notational convenience:

$$\mathbb{F}_2 = C^{-1}(X) \xrightarrow{\delta} C^0(X) \xrightarrow{\delta} \cdots \xrightarrow{\delta} C^{\dim X}(X).$$

We remark that $\partial^2 = 0$ and $\delta^2 = 0$. We also define the usual objects of cohomology theory:

- $B^k(X) = \operatorname{im}(\delta : C^{k-1}(X) \to C^k(X))$
- $Z^k(X) = \ker(\delta : C^k(X) \to C^{k+1}(X))$
- $H^k(X) = Z^k(X)/B^k(X)$.

See Chapter 7, Section 1 for details on the theory of cohomology of graphs.

Define the *Hamming norm* of $c \in C^k(X)$ by

$$|c| = |\{\sigma \in X_k : c(\sigma) = 1\}|.$$

The *normalized Hamming norm* of c is

$$\|c\| = \frac{|c|}{|X_k|}.$$

We now introduce three geometric notions.

DEFINITION 1.4. Let $\eta, \mu, L > 0$. X satisfies an *L-coisoperimetric inequality in dimension k* if, for all $b \in B^k(X)$, there exists an $a \in C^{k-1}(X)$ such that $a = \delta b$ and $\|a\| \leq L\|b\|$. Equivalently, X satisfies an L-coisoperimetric inequality if and only if every $a \in C^{k-1}$ satisfies the inequality

$$\|\delta a\| \geq \frac{1}{L} \min\left\{\|a + z\| : z \in Z^{k-1}(X)\right\}.$$

We say that a *co-fills* b if $a = \delta b$.

REMARK. In dimension 1, L-coisoperimetry is equivalent to $\frac{1}{L}$-edge expansion on each connected component.

DEFINITION 1.5. Let $\eta, \mu, L > 0$. X has *η-coboundary expansion in dimension k* if, for each $a \in C^{k-1}(X)$, the following inequality holds:

$$\|\delta a\| \geq \eta \min\{\|a + b\| : b \in B^{k-1}(X)\}.$$

REMARK. In dimension 1, η-coboundary expansion is equivalent to η-edge expansion.

We remark that X has η-coboundary expansion in dimension k if and only if X satisfies a $\frac{1}{\delta}$-coisoperimetric inequality in dimension k and $H^{k-1}(X) = 0$. Indeed, $a \notin B^{k-1}(X)$ if and only if

$$\|[a]\| = \min\{\|a + b\| : b \in B^{k-1}(X)\} > 0.$$

DEFINITION 1.6. Fix $\mu > 0$ and $k \in \mathbb{N}$. X is said to satisfy the *μ-cosystolic inequality in dimension k* if, for all $z \in Z^k(X)/B^k(X)$, we have the inequality

$$\|z\| \geq \mu.$$

REMARK. In dimension 0, μ-cosystolic inequality is equivalent to the fact that every connected component of X has size at least $\mu|X_0|$.

These definitions generalize to arbitrary cell complexes. For intuition, we assume that X is a cell decomposition of a d-dimensional PL manifold without boundary M. Letting Y denote the dual cell complex, we see that there is a one-to-one correspondence between X_k and Y_{d-k}. We then have the following commutative diagram:

$$\begin{array}{ccc} C^{k-1}(X) & \xrightarrow{\delta} & C^k(X) \\ \downarrow \cong & & \downarrow \cong \\ C_{d-k+1}(Y) & \xrightarrow{\partial} & C_{d-k}(Y) \end{array}$$

For example, for the 1-dimensional case (edges), the expansion / coisoperimetry of X correspond to the top-dimensional isoperimetry in Y. For the 2-dimensional case, the expansion / coisoperimetry of X corresponds to codimension-1 isoperimetry in Y.

We now state the main result of this chapter. To this end, we shall make use of the following technical condition[1]:

DEFINITION 1.7. X is locally ε-sparse if, for each $k \in \{1, \ldots, \dim X\}$ and every $v \in X_0$, the following inequality holds:

$$|\{\sigma \in X_k : v \in \sigma\}| \leq \varepsilon |X_k|.$$

THEOREM 1.8 (Gromov's topological overlap theorem). *Let X be a finite simplicial complex with underlying space $\|X\|$ that is locally ε-sparse for some $\varepsilon > 0$. Let $L, \mu > 0$ and $d \in \mathbb{N}$ and fix $\varepsilon > 0$ that is sufficiently small with respect to L, μ, and d. Suppose that X has L-isoperimetry in dimensions $k = 1, \ldots, d$ and satisfies the μ-cosystolic inequality in dimensions $k = 1, \ldots, d-1$. Then, for each continuous map $f : \|X\| \to \mathbb{R}^d$, there exists $p \in \mathbb{R}^d$ such that*

$$|\{\sigma \in X_d : f(\sigma) \ni p\}| \geq \lambda |X_d|,$$

where $\lambda = \lambda(L, M, d, \varepsilon) > 0$. In this case, we say that X has topological λ-overlap *for maps into $\mathbb{R}^d$.*

An analogous problem is to consider only linear maps—this goes by the name of *geometric overlap*. We will not discuss this topic, however.

2. Outline of the Proof (Day 3, Period 3)

The outline of the proof is as follows:

(1) Without loss of generality, f is a piecewise-linear map in "general position."
(2) Observe that $\mathbb{R}^d \cong \mathbb{R}^d \times \{1\} \subseteq \mathbb{R}^{d+1}$ View f as a piecewise-linear map in "general position", so that $f : \|X\| \to \partial\Delta^{d+1} \cong \mathbb{S}^d$. Then take a "sufficiently fine" triangulation T of $\partial\Delta^{d+1}$ in "general position" with respect to f.
(3) Define, for every k-simplex τ of T and every $(d-k)$-simplex σ of X, $f^\sharp(\tau)$ to be the *algebraic intersection* number between $f(\sigma)$ and τ: this is given by

$$|f(\sigma) \cap \tau| \bmod 2$$

if $\dim \tau > 0$ and by

$$|f^{-1}(\tau) \cap \sigma| \bmod 2$$

if $\dim \tau = 0$. We can then extend linear to obtain $f^\sharp : C_k(T) \to C^{d-k}(X)$. The key fact is that the following diagram commutes:

$$\begin{array}{ccc} C_k(T) & \xrightarrow{f^\sharp} & C^{d-k}(X) \\ \big\downarrow \partial & & \big\downarrow \delta \\ C_{k-1}(T) & \xrightarrow{f^\sharp} & C^{d-k+1}(X). \end{array}$$

(4) We need to make the notion of "sufficiently fine" precise: for each $k > 0$ and every $\tau \in X_k$,

$$\|f^\sharp(\tau)\| < d^2 \varepsilon |X_{d-k}|.$$

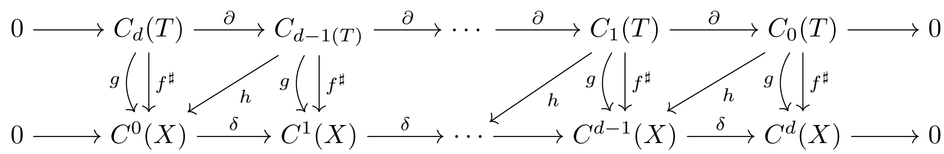
[1] Wagner: "I am not sure to what extent this condition is needed."

3. Proof (Day 4, Period 2)

LEMMA 3.1. *Let* $[T] = \sum_{\tau \in T_d} \tau$. *Then* $f^\sharp([T]) = \mathbf{1} \in C^0(X)$.

LEMMA 3.2. *There exists a* $v \in T_0$ *such that* $\|f^\sharp(v)\| \geq \lambda$.

We now establish the following commutative diagram:



To this end, we begin by observing that $f^\sharp$ is a chain-cochain map (Section 2, Step 3). Define another chain-cochain map $g : C_k(T) \to C^{d-k}(X)$ by setting $g(c) = 0$ for all $c \in C_k(T)$. Observe the following:

LEMMA 3.3. *It follows from Lemma 3.2 that there exists a map* $h : C_k(T) \to C^{d-k-1}(X)$ *such that*

$$f^\sharp(c) = \delta h(c) + h(\partial c)$$

for all $c \in C_k(T)$ *and that* $\|h(\tau)\| \leq S_k$ *for all* $\tau \in T_k$.

Suppose Lemma 3.3 is true and Lemma 3.2 is false. Consider

$$\mathbf{1} = f^\sharp([T]) = \delta(h[T]) + h(\partial[T]),$$

but the right-hand side is 0.

We now construct h by induction on k. What we need is that, for $\tau \in T_k$, $\delta h(\tau) = f^\sharp(\tau) + h(\partial\tau)$.

For $k = -1$, we set $h = 0$.

For $k = 0$, we consider $v \in T_0$. Observe that

$$f^\sharp(v) = f^\sharp(v) - f^\sharp(v_0) = f^\sharp(v - v_0) = f^\sharp(\partial c) \in B^d(X). = \delta f^\sharp(c),$$

provided that there exists a 1-chain $c \in C_1(T)$ such that $\partial c = v - v_0$. Existence follows from connectedness of S^d. We now choose $h(v) \in C^{d-1}(X)$ such that $\delta h(v) = f^\sharp(v)$ and $\|h(v)\| < L\|f^\sharp(v)\| < \delta$. We can set $S_0 = L\lambda$.

We now let $k > 0$ and assume that h is defined on $C_{k-1}(T)$. Consider $\tau \in T_k$. $f^\sharp(\tau)$ is defined, and $h(\partial\tau)$ is defined. We also have that $\|f^\sharp(\tau)\| \leq \varepsilon d^2$ and that $\|h(\partial\tau)\| \leq (k+1)S_{k-1}$. Observe that

$$\begin{aligned} \delta(f^\sharp(\tau) - h(\partial\tau)) &= \delta f^\sharp(\tau) - \delta h(\partial\tau) \\ &= f^\sharp(\partial\tau) - \delta h(\partial\tau) \\ &= f^\sharp(\partial\tau) - h(\partial\partial\tau) \\ &= 0. \end{aligned}$$

Therefore, $z = f^\sharp(\tau) - h(\partial\tau) \in Z^{d-k}(X)$. If $z \notin B^{d-k}(X)$, then $M \leq \|z\| \leq d^2\varepsilon + (k+1)S_{k-1}$. Else, we choose $h(v)$ such that $\delta h(v) = z = f^\sharp(v) - h(\partial v)$. In this case,

$$\|h(v)\| \leq L(d^2\varepsilon + (k+1)S_{k-1}) := S_k.$$

We now observe that

$$S_0 = L\lambda$$
$$S_k = d^2\varepsilon(L + L^2 + \cdots + L^k) + (k+1)!L^{k+1}\lambda \qquad < M \text{ for suitable } \lambda.$$

We can now set

$$\lambda = \min \frac{1}{(k+1)!L^{l+1}}.$$